State Space Modeling and Analysis

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Basic Concepts related to State Space

- **State Variables**

  The smallest possible subset of system variables that can represent the entire state of the system at any given time

- **State Vector**

  All the state variables $\{q_1, q_2, \ldots, q_n\}$ can be looked on as components of state vector.

- **State Space**

  A space whose coordinates consist of state variables is called a state space. Any state can be represented by a point in state space.

State Space Representation

Introduction:

- Two parts:
  - A set of first order ODEs that represents the derivative of each state variable x_i as an algebraic function of the set of state variables $\{x_i\}$ and the inputs $\{u_i\}$.

$$\begin{cases} \dot{q}_1 = f_1(x_1, x_2, x_3, \ldots, x_n, u_1, u_2, u_3, \ldots, u_m) \\ \dot{q}_2 = f_2(x_1, x_2, x_3, \ldots, x_n, u_1, u_2, u_3, \ldots, u_m) \\ \vdots \\ \dot{q}_n = f_n(x_1, x_2, x_3, \ldots, x_n, u_1, u_2, u_3, \ldots, u_m) \end{cases}$$

$$\dot{x} = \mathbf{A}x + \mathbf{B}u$$

  - A set of equations that represents the output variables as algebraic functions of the set of state variables $\{x_i\}$ and the inputs $\{u_i\}$.

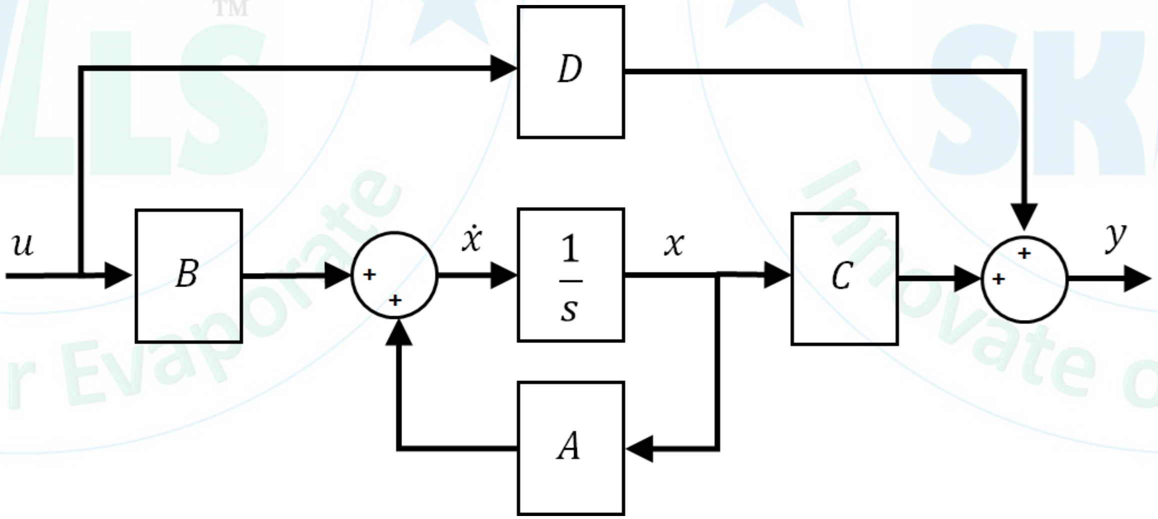
$$\begin{cases} y_1 = f_1(x_1, x_2, x_3, \ldots, x_n, u_1, u_2, u_3, \ldots, u_m) \\ y_2 = f_2(x_1, x_2, x_3, \ldots, x_n, u_1, u_2, u_3, \ldots, u_m) \\ \vdots \\ y_n = f_n(x_1, x_2, x_3, \ldots, x_n, u_1, u_2, u_3, \ldots, u_m) \end{cases}$$

$$y = \mathbf{C}x + \mathbf{D}u$$

Differential Equations and State Space Representation

- **The state equations are:**

$$\dot{\mathbf{X}} = \mathbf{A}X + \mathbf{B}u$$
$$y = \mathbf{C}X + \mathbf{D}u$$



Differential Equations and State Space Representation

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_0 u(t)$$

- **A second-order differential equation requires two state variables:**

$$x_1(t) = y(t) \qquad x_2(t) = \dot{y}(t)$$

- **We can reformulate the differential equation as a set of three equations:**

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -a_0 x_1(t) - a_1 x_2(t) + b_0 u(t)$$
$$y(t) = x_1(t)$$

- **We can write these in matrix form as:**

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- **This can be extended to an N^{th}-order differential equation of this type:** $y^{(N)}(t) + \sum_{i=0}^{N-1} a_i y^{(i)}(t) = b_0 u(t)$

- **The state variables are defined as:** $x_i(t) = y^{(i-1)}(t), \quad i = 1, 2, \ldots, N$

Differential Equations and State Space Representation

- **The resulting state equations are:**

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t)\\
\dot{x}_2(t) &= x_3(t)\\
&\vdots\\
\dot{x}_{N-1}(t) &= x_N(t)\\
\dot{x}_N(t) &= -\sum_{i=0}^{N-1} a_i x_{i+1}(t) + b_0 u(t)\\
y(t) &= x_1(t)
\end{aligned}
$$

$$
\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-1} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}
$$

$$
\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \mathbf{D} = 0
$$

$$
\begin{aligned}
\dot{\mathrm{X}} &= \mathbf{A}X + \mathbf{B}u\\
y &= \mathbf{C}X + \mathbf{D}u
\end{aligned}
$$

Differential Equations and State Space Representation

- **Next, consider a differential equation with a more complex forcing function:**

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_1\dot{v}(t) + b_0 u(t)$$

- **The state model is:**

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t) \qquad y(t) = \begin{bmatrix}b_0 & b_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

$$\dot{\mathrm{X}} = \mathbf{A}X + \mathbf{B}u$$
$$y = \mathbf{C}X + \mathbf{D}u$$

- **We can verify this by expanding the matrix equation:**

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -a_0x_1(t) - a_1x_2(t) + u(t)$$
$$y(t) = b_0x_1(t) + b_1x_2(t)$$

- **To construct the original equation, differentiate the last equation:**

$$\begin{aligned}\dot{y}(t) &= b_0\dot{x}_1(t) + b_1\dot{x}_2(t)\\ &= b_0x_2(t) + b_1[-a_0x_1(t) - a_1x_2(t) + u(t)]\\ &= -a_1y(t) + (a_1b_0 - a_0b_1)x_1(t) + b_0x_2(t)\end{aligned}$$

Differential Equations and State Space Representation

- Differentiate the last equation again and substitute:

$$
\begin{aligned}
\ddot{y}(t) &= -a_1\dot{y}(t) + (a_1 b_0 - a_0 b_1)\dot{x}_1(t) + b_0\dot{x}_2(t) + b_1\dot{u}(t) \\
&= -a_1\dot{y}(t) + (a_1 b_0 - a_0 b_1)x_2(t) \\
&\quad + b_0[-a_0 x_1(t) - a_1 x_2(t) + v(t)] + b_1\dot{u}(t) \\
&= -a_1\dot{y}(t) - b_0 a_0 x_1(t) - a_0 b_1 x_2(t) + b_0 u(t) + b_1\dot{u}(t) \\
&= -a_1\dot{y}(t) - a_0(b_0 x_1(t) + b_1 x_2(t)) + b_0 u(t) + b_1\dot{u}(t) \\
&= -a_1\dot{y}(t) - a_0 y(t) + b_0 u(t) + b_1\dot{u}(t)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathrm{X}} &= \mathbf{A}X + \mathbf{B}u \\
y &= \mathbf{C}X + \mathbf{D}u
\end{aligned}
$$

- Hence, given a general LTI system:

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_0 u(t) + b_1\dot{u}(t)$$

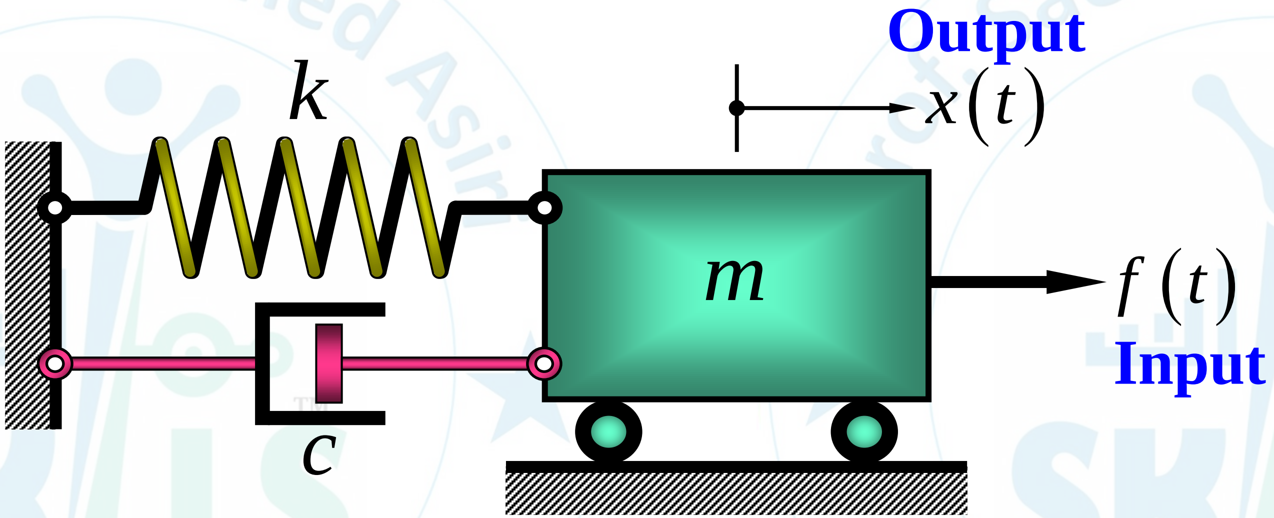
$$y^{(N)}(t) + \sum_{i=0}^{N-1} a_i y^{(i)}(t) = \sum_{i=0}^{N-1} b_i u^{(i)}(t)$$

$$
\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-1} \end{bmatrix} \quad
\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
$$

$$\mathbf{C} = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_N \end{bmatrix} \quad \mathbf{D} = 0$$

State Space Representation

Introduction:



$$m\ddot{x} + c\dot{x} + kx = F$$

F = Control force

State Space Representation

Introduction:

Let $x_1 = x, \quad x_2 = \dot{x}$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\mathbf{X}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{X}} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{\mathbf{B}} \underbrace{F}_{u}$$

State Space Representation

Introduction:

$$\dot{X} = \mathbf{A}X + \mathbf{B}u$$

where

$$\mathbf{X} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \quad u = F$$

State Space Representation

Introduction:

$$\dot{X} = \mathbf{A}X + \mathbf{B}u$$
$$y = \mathbf{C}X + \mathbf{D}u$$

where

X = state vector, **y** = Output vector, *u* = control vector

A = system matrix, **B** = Input matrix

C = Output matrix, **D** = Feedforward Matrix

State Space Representation

Introduction:

Output $y = x$

$$y = \mathbf{C}X + \mathbf{D}u$$

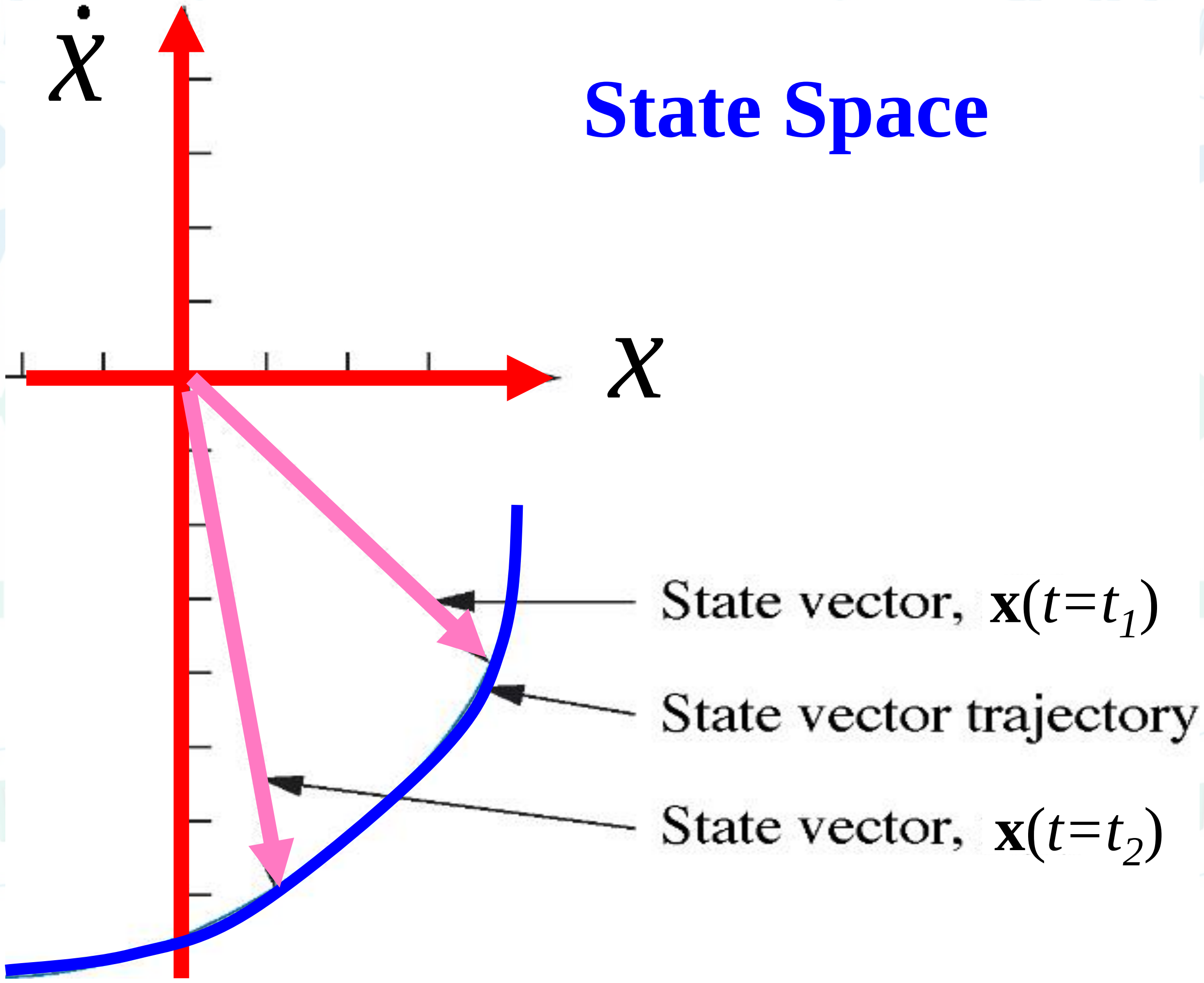
where

$$\mathbf{X} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \; C = [1 \quad 0], \; D = 0, \; u = F$$

Displacement Sensor

State Space Representation

Introduction:

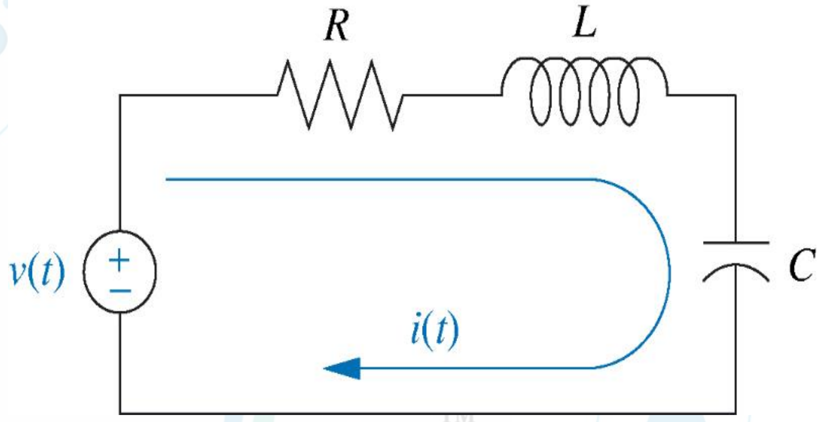


State Space Representation

Introduction:

System Equation $$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt=v$$

Output Equation $$v_L=L\frac{di}{dt}=-Ri-\frac{1}{C}\int idt+v$$



$$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt=v$$ **Let** $$\frac{dq}{dt}=i \Rightarrow L\frac{di}{dt}+Ri+\frac{1}{C}q=v$$

$$L\frac{di}{dt}=-\frac{1}{C}q-Ri+v$$

$$\frac{di}{dt}=-\frac{1}{LC}q-\frac{R}{L}i+\frac{1}{L}v$$

State Space Representation

Introduction:

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{B}u$$

$$\frac{dq}{dt} = i$$

$$\frac{di}{dt} = -\frac{1}{LC}q - \frac{R}{L}i + \frac{1}{L}v$$

where $\mathbf{X} = \begin{bmatrix} q \\ i \end{bmatrix}, \dot{\mathbf{X}} = \begin{bmatrix} \frac{dq}{dt} \\ \frac{di}{dt} \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix},$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}, \quad u = v$$

State Space Representation

Introduction:

Output Equation

$$v_L = L\frac{di}{dt} = -Ri - \frac{1}{C}\int idt + v$$

$$v_L = -\frac{1}{C}q - Ri + v$$

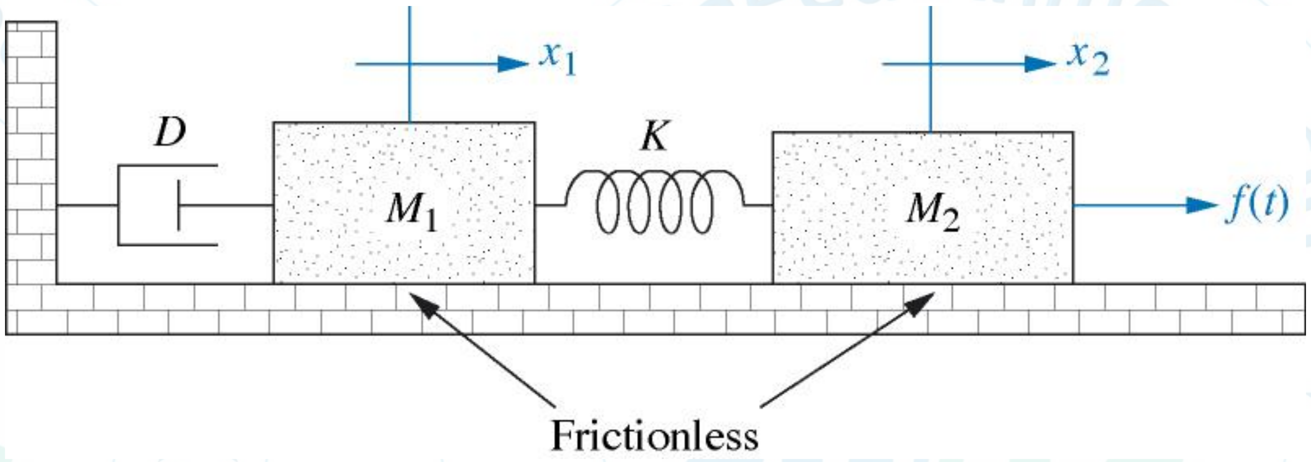
$$\mathbf{y} = C\mathbf{X} + Du$$

where

$$\mathbf{X} = \begin{bmatrix} q \\ i \end{bmatrix}, C = [-1/C \ \ -R], D = 1, u = v$$

State Space Representation

Example:



Equations of Motion

$$M_1\ddot{x}_1 + D\dot{x}_1 + Kx_1 - Kx_2 = 0$$

$$-Kx_1 + M_2\ddot{x}_2 + 0\dot{x}_2 + Kx_2 = f$$

State Space Representation

Example:

Let $X = \begin{bmatrix} x_1 & \dot{x}_1 & x_2 & \dot{x}_2 \end{bmatrix}^T$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix}}_{\mathbf{B}} f$$

State Space Representation

Example:

$$y = x_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + 0 \times f$$

Displacement Sensor (x_1)

$$= C\,X + D\,u$$

where C =[1 0 0 0] and D = 0

Transfer Function to State Space

System Equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

Let

$$x_1 = y,\ x_2 = \frac{dy}{dt},\ x_3 = \frac{d^2 y}{dt^2},\ \ldots.,\ x_n = \frac{d^{n-1} y}{dt^{n-1}}$$

$$\Rightarrow \dot{x}_1 = \frac{dy}{dt},\ \dot{x}_2 = \frac{d^2 y}{dt^2},\ \dot{x}_3 = \frac{d^3 y}{dt^3},\ \ldots.,\ \dot{x}_n = \frac{d^n y}{dt^n}$$

or

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \ldots.,\ \dot{x}_{n-1} = x_n,$$

$$\dot{x}_n = -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n + b_0 u$$

Transfer Function to State Space

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \cdot\cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \dots & -a_{n-1} \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot\cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot\cdot \\ 0 \\ b_0 \end{bmatrix}}_{\mathbf{B}} u$$

$$y = \underbrace{[1 \quad 0 \quad 0 \quad \dots \quad 0]}_{\mathbf{C}} [x_1 \quad x_2 \quad x_3 \quad \dots \quad x_{n-1} \quad x_n]^T$$

Transfer Function to State Space

Example:

$$\frac{C}{R} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

Let $x_1 = c,\ x_2 = \dot{c},\ x_3 = \ddot{c}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r, \\ y = c = x_1 \end{cases}$$

Transfer Function to State Space

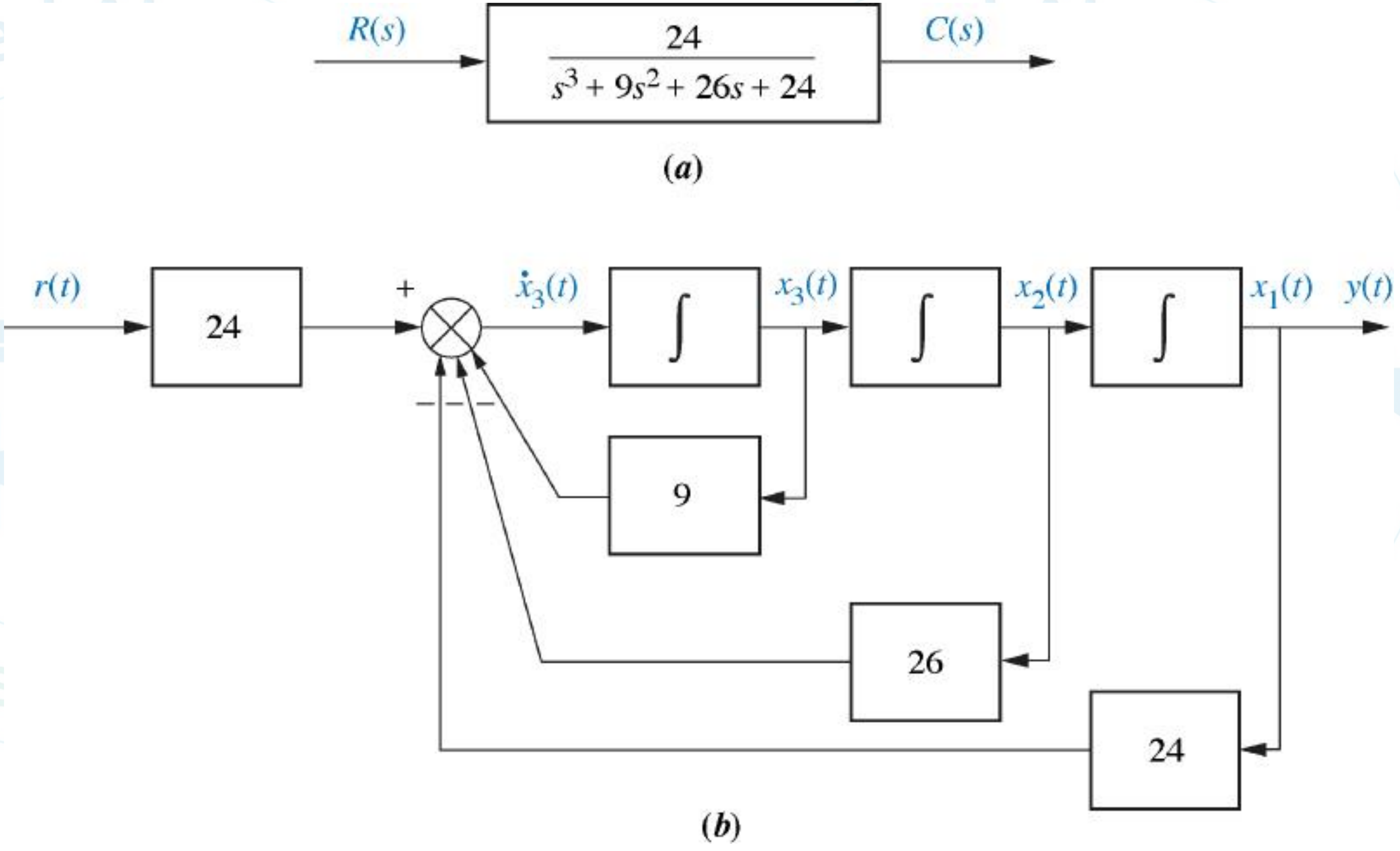
Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix}}_{\mathbf{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Transfer Function to State Space

Example:



Transfer Function to State Space

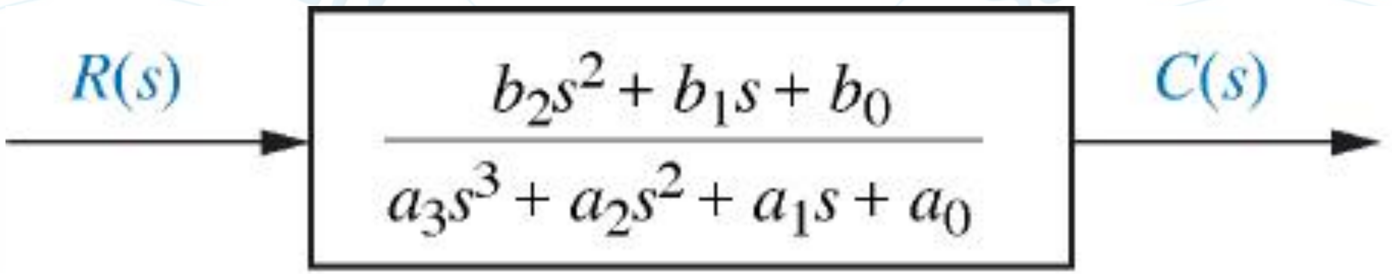
Example:

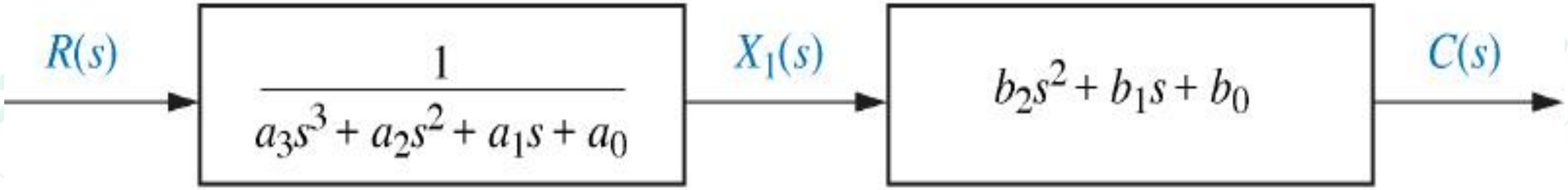
MATLAB

```
>> n=24;
>> d=[1  9  26  24];
>>
[A,B,C,D]=tf2ss(n,d)
```

```
A =
    -9   -26   -24
     1     0     0
     0     1     0

B =
     1
     0
     0

C =      0     0    24

D =      0
```

T. Fn. with numerator Poly. to S-S

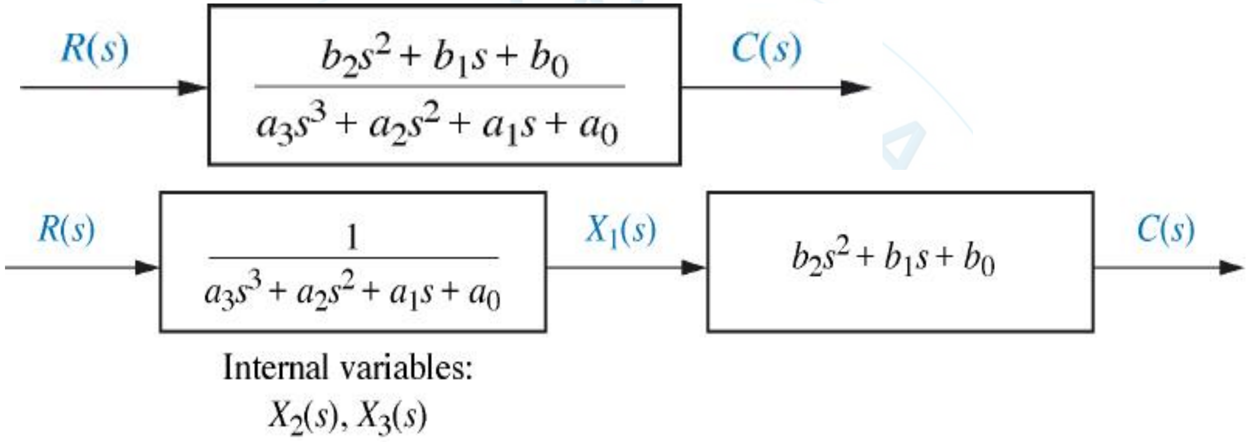


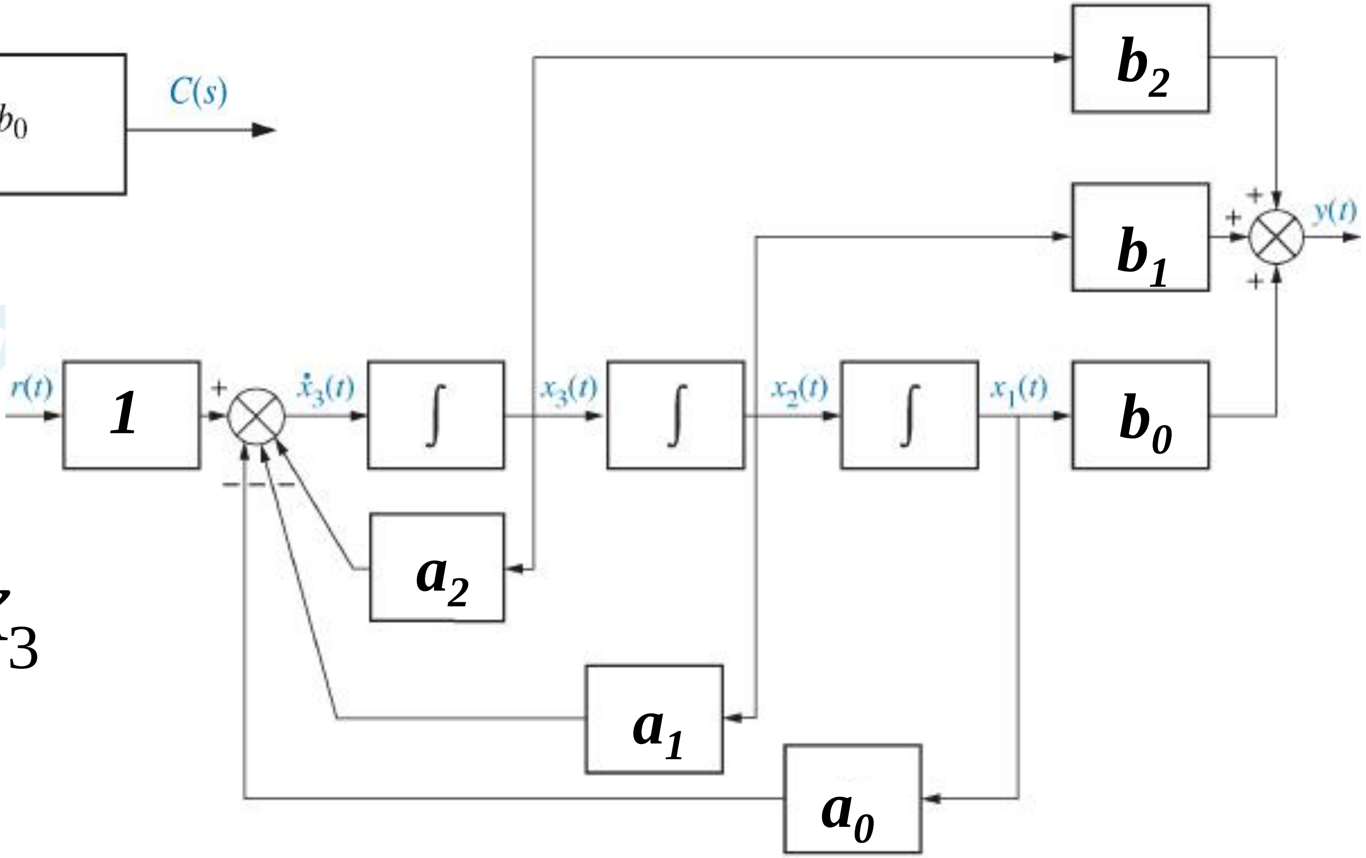


Internal variables:

$X_2(s), X_3(s)$

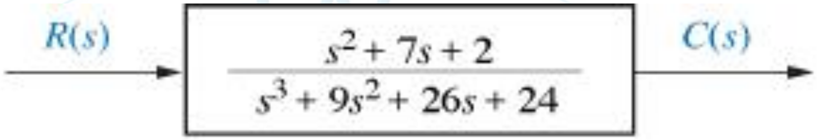
T. Fn. with numerator Poly. to S-S

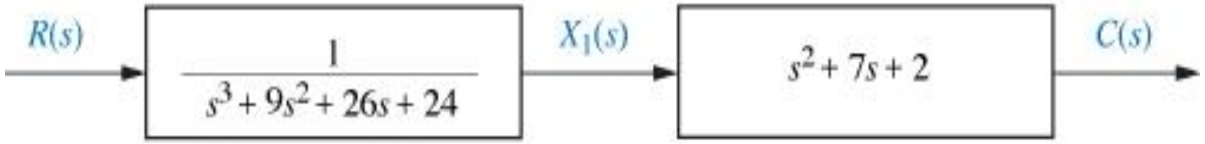




$$x_1 = x_1,\ \dot{x}_1 = x_2,\ \ddot{x}_1 = x_3$$

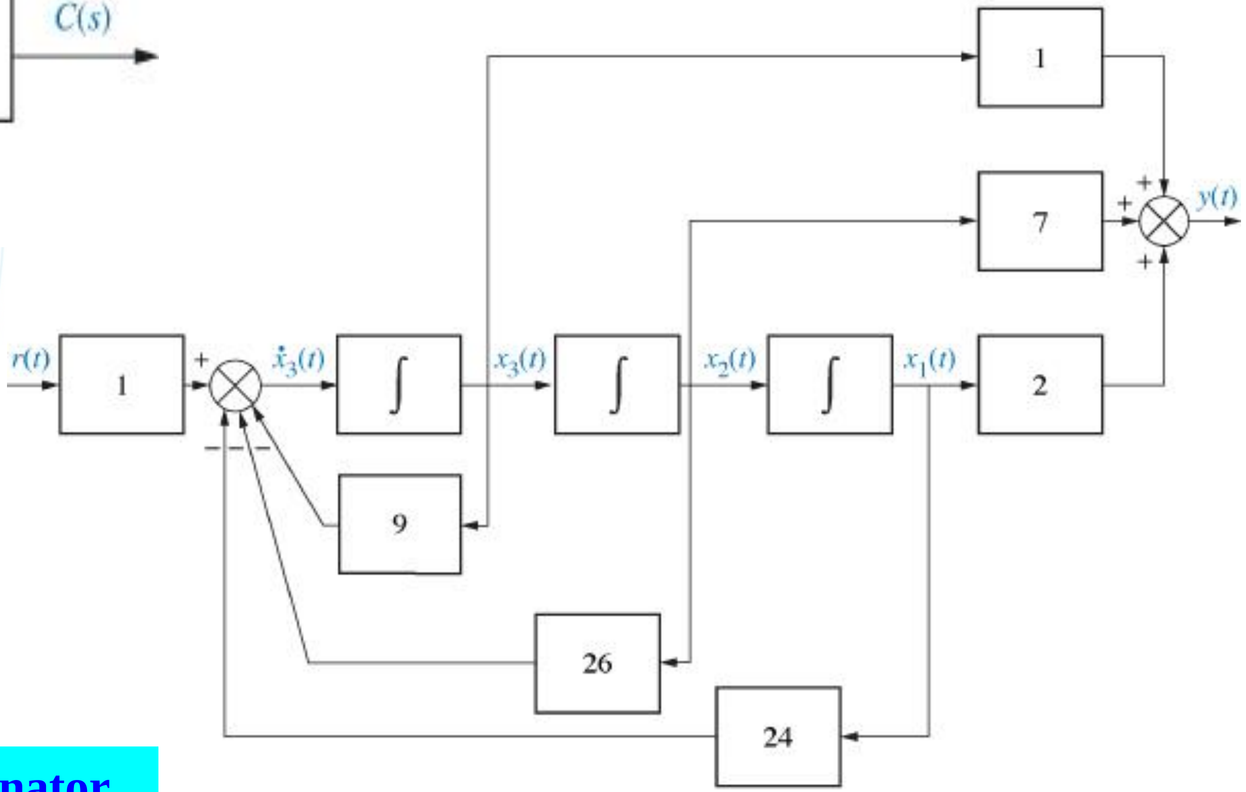
T. Fn. with numerator Poly. to S-S





Internal variables:
$X_2(s)$, $X_3(s)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u$$



Same as Transfer function with polynomial denominator

T. Fn. with numerator Poly. to S-S

But $C(s) = \left(b_2 s^2 + b_1 s + b_0\right) X_1(s) = \left(s^2 + 7s + 2\right) X_1(s)$

$\Rightarrow \quad c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1$

Let $x_1 = x_1,\ \dot{x}_1 = x_2,\ \ddot{x}_1 = x_3$

$\Rightarrow \quad y = c = x_3 + 7x_2 + 2x_1 = b_2 x_3 + b_1 x_2 + b_0 x_1$

or $y = [b_0 \quad b_1 \quad b_2][x_1 \quad x_2 \quad x_3]^T = \underbrace{[2 \quad 7 \quad 1]}_{C}[x_1 \quad x_2 \quad x_3]^T$

T. Fn. with numerator Poly. to S-S

```
>> n=[1  7  2];
>> d=[1  9  26  24];
>> [A,B,C,D]=tf2ss(n,d)

A =
   -9   -26   -24
    1     0     0
    0     1     0

B =
    1
    0
    0

C =      1     7     2

D =      0
```

T. Fn. with numerator Poly. to S-S

SUMMARY

- Denominator of Tr. Fn. (**Poles**) affects the ***A*** matrix
- Numerator of Tr. Fn. (**Zeros**) affects the ***C*** matrix
- Numerator **gain** affects ***B*** matrix

State-Space to Transfer Function

State-Space Model

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}u,$$
$$\mathbf{y} = \mathbf{C}\mathbf{X} + \mathbf{D}u$$

Laplace Transform

$$s\mathbf{X} = \mathbf{A}\mathbf{X} + \mathbf{B}u,$$
$$\mathbf{y} = \mathbf{C}\mathbf{X} + \mathbf{D}u$$

$$\Rightarrow \quad \mathbf{X} = \left(sI - \mathbf{A}\right)^{-1}\mathbf{B}u$$

or

$$\mathbf{Y}(s) = \left[\mathbf{C}\left(sI - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]u$$

State-Space to Transfer Function

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State-Space to Transfer Function

Example

$$(s\mathbf{I}-\mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{adj(sI-A)}{\det(sI-A)} = \frac{\begin{bmatrix} (s^2+3s+2) & (s+3) & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

$$\mathbf{C} = [1 \quad 0 \quad 0\,], \quad \mathbf{D} = 0$$

State-Space to Transfer Function

Example

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{adj(sI-A)}{\det(sI-A)}=\frac{\begin{bmatrix}(s^2+3s+2) & (s+3) & 1\\ -1 & s(s+3) & s\\ -s & -(2s+1) & s^2\end{bmatrix}}{s^3+3s^2+2s+1}$$

$$\mathbf{B}=\begin{bmatrix}10 & 0 & 0\end{bmatrix}^T,\ \mathbf{C}=[1\quad 0\quad 0\ ],\ \mathbf{D}=0$$

$$\mathbf{Y}(s)=\left[\mathbf{C}(sI-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]u=\frac{10(s^2+3s+2)}{(s^3+3s^2+2s+1)}u$$

Transfer Function

State-Space to Transfer Function

Example

Method 1

```
>> syms s
>> A=[0  1  0;0  0  1;-1  -2, -3];
>> B=[10  0  0]';
>> C=[1  0  0];
>> AS=s*eye(3)-A;
>> ASI=inv(AS);
>> G=C*ASI*B
```

G = 10*(s^2+3*s+2)/(s^3+3*s^2+2*s+1)

State-Space to Transfer Function

Example

Method 2

```
>> A=[0  1  0;0  0  1;-1  -2, -3];
>> B=[10  0  0]';
>> C=[1  0  0];
>> D=0;
>> [n,d]=ss2tf(A,B,C,D)

n =         0   10.0000   30.0000   20.0000

d =    1.0000    3.0000    2.0000    1.0000

>> G=tf(n,d)
```

```
Transfer function:   10 s^2 + 30 s + 20
                     ----------------------------
                      s^3 + 3 s^2 + 2 s + 1
```

State Space Representation

Obtaining State Space Representation

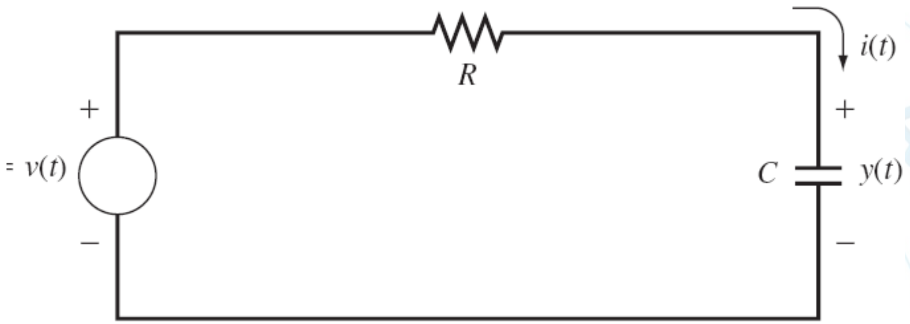
- Identify State Variables
  - *Rule of Thumb:*
    - *Nth order ODE requires N state variables.*
    - *Position and velocity of inertia elements are natural state variables for translational mechanical systems.*
- Eliminate all algebraic equations written in the modeling process.
- Express the resulting differential equations in terms of state variables and inputs in coupled first order ODEs.
- Express the output variables as algebraic functions of the state variables and inputs.
- For linear systems, put the equations in matrix form.

$$\dot{\mathbf{x}} = \mathbf{A}\cdot\underbrace{\mathbf{x}}_{\text{State Vector}} + \mathbf{B}\cdot\underbrace{\mathbf{u}}_{\text{Input Vector}}$$

$$\underbrace{\mathbf{y}}_{\text{Output Vector}} = \mathbf{C}\cdot\mathbf{x} + \mathbf{D}\cdot\mathbf{u}$$

State Space Representation

Electrical system Example:



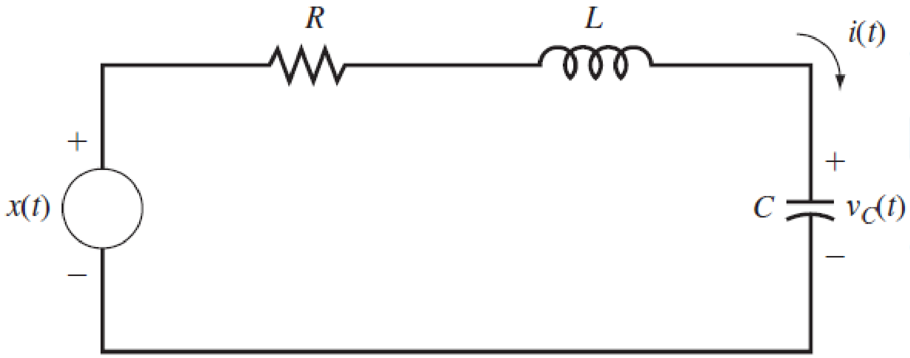
$$\frac{dy(t)}{dt}+\frac{1}{RC}y(t)=\frac{1}{RC}v(t)$$

$$\mathbf{A}=[-a_0]=\left[-\frac{1}{RC}\right]$$

$$\mathbf{B}=[b_0]=\left[\frac{1}{RC}\right]\quad \mathbf{C}=[1]\quad \mathbf{D}=0$$

$$[\dot{x}_1(t)]=\left[-\frac{1}{RC}\right][x_1(t)]+\left[\frac{1}{RC}\right]v(t)$$

$$y(t)=[1]\,[x_1(t)]$$



$$\frac{d^2y(t)}{dt^2}+\left(\frac{R}{L}\right)\frac{dy(t)}{dt}+\frac{1}{LC}y(t)=\frac{1}{LC}v(t)$$

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1/LC & -R/L\end{bmatrix}$$

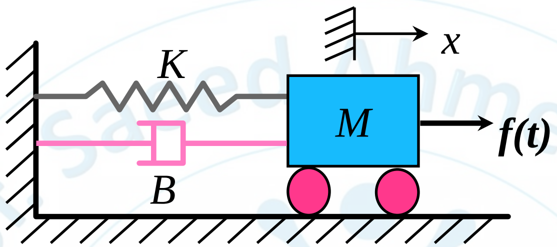
$$\mathbf{B}=\begin{bmatrix}0\\ b_0\end{bmatrix}=\begin{bmatrix}0\\ 1/LC\end{bmatrix}\quad \mathbf{C}=[1\quad 0]\quad \mathbf{D}=0$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1/LC & -R/L\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1/LC\end{bmatrix}v(t)$$

$$y(t)=[1\quad 0]\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

State Space Representation

Mechanical system Example:



EOM: $M\ddot{x} + B\dot{x} + Kx = f(t)$

Q: What information about the mass do we need to know to be able to solve for x(t) for $t \geq t_0$?

Input: $f(t), \quad t \geq t_0$

Initial Conditions (ICs): $x(t_0)$ $\qquad q_1 = x(t)$

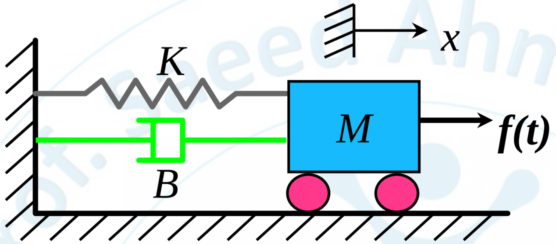
$\dot{x}(t_0)$ $\qquad q_2 = \dot{x}(t)$

Rule of Thumb

Number of state variables = Sum of orders of EOMs

State Space Representation

Mechanical system Example:



EOM $M\ddot{x} + B\dot{x} + Kx = f(t)$

State Variables: $q_1 = x,$
$q_2 = \dot{x}$

Outputs: $y_1 = x, \quad y_2 = -B\dot{x}$

State Space Representation:

State equation

$$\begin{cases} \dot{q}_1 = \dot{x} = q_2 \\ \dot{q}_2 = \ddot{x} = \dfrac{1}{M}(-Bq_2 - Kq_1) + \dfrac{1}{M}f(t) \end{cases}$$

Output equation

$$\begin{cases} y_1 = x = q_1 \\ y_2 = -B\dot{x} = -Bq_2 \end{cases}$$

Matrix Form

$$\underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\dfrac{K}{M} & -\dfrac{B}{M} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix}}_{\mathbf{B}} \underbrace{[f]}_{\mathrm{u}}$$

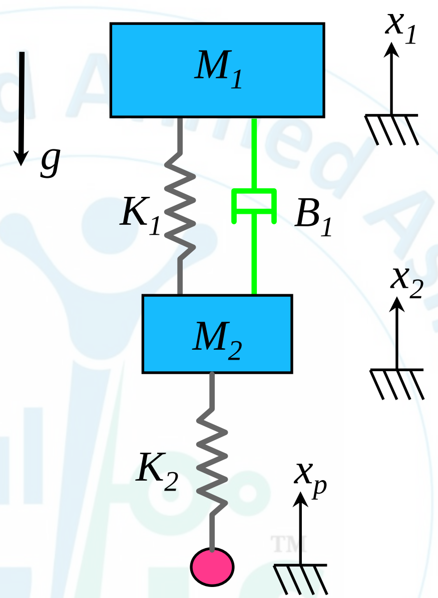
$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -B \end{bmatrix}}_{\mathbf{C}} \cdot \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{0_{2\times1}}_{\mathrm{D}} \cdot \mathrm{u}$$

State Space Representation

Exercise

Represent the 2 DOF suspension system in a state space representation.

Let the system output be the relative position of mass M_1 with respect to M_2 .



$$M_1\ddot{x}_1 + B_1\dot{x}_1 - B_1\dot{x}_2 + K_1x_1 - K_1x_2 = 0$$
$$M_2\ddot{x}_2 - B_1\dot{x}_1 + B_1\dot{x}_2 - K_1x_1 + (K_1 + K_2)x_2 = K_2x_p$$

$$q_1 = x_1, \quad q_2 = \dot{x}_1, \quad q_3 = x_2, \quad q_4 = \dot{x}_2$$

State Variables:

Output: $y = x_1 - x_2$ **Input:** x_p

State Space Representation:

$$\underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_1}{M_1} & -\frac{B_1}{M_1} & \frac{K_1}{M_1} & \frac{B_1}{M_1} \\ 0 & 0 & 0 & 1 \\ \frac{K_1}{M_2} & \frac{B_1}{M_2} & -\frac{K_1+K_2}{M_2} & -\frac{B_1}{M_2} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_2}{M_2} \end{bmatrix}}_{\mathbf{B}} \underbrace{x_p}_{\mathrm{u}}$$

$$\mathrm{y} = \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}}_{\mathbf{C}} \cdot \mathbf{x} + \underbrace{0}_{\mathrm{D}} \cdot \mathrm{u}$$

Input/Output Representation

Input/Output Model

Uses one nth order ODE to represent the relationship between the input variable, u(t), and the output variable, y(t), of a system.

For linear time-invariant (LTI) systems, it can be represented by :

where

$$a_n y^{(n)} + \cdots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_m u^{(m)} + \cdots + b_2 \ddot{u} + b_1 \dot{u} + b_0 u(t)$$

- To solve an input/output differential equation, we need to know

Input: $u(t), \quad t \geq 0$

Initial Conditions (ICs): $\underbrace{y(0), \quad \dot{y}(0), \quad \cdots, \quad y^{(n-1)}(0)}_{n}$

- To obtain I/O models:
  - Identify input/output variables.
  - Derive equations of motion.
  - Combine equations of motion by eliminating all variables except the input and output variables and their derivatives.

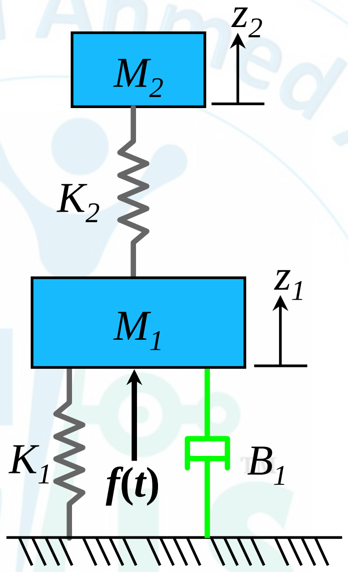
Input/Output Representation

Example

Vibration Absorber

EOM:

$$M_1\ddot{z}_1 + B_1\dot{z}_1 + (K_1 + K_2)z_1 - K_2 z_2 = f(t)$$
$$M_2\ddot{z}_2 + K_2 z_2 - K_2 z_1 = 0$$



- Find input/output representation between input $f(t)$ and output z_2.
  - Need to eliminate z_1 and its time derivative of all orders

$$z_1 = \frac{1}{K_2}\{M_2\ddot{z}_2 + K_2 z_2\}$$

$$\dot{z}_1 = \frac{1}{K_2}\left\{M_2 z_2^{(3)} + K_2\dot{z}_2\right\},$$

$$\ddot{z}_1 = \frac{1}{K_2}\left\{M_2 z_2^{(4)} + K_2\ddot{z}_2\right\}$$

$$z_2^{(4)} + \frac{B_1}{M_1}z_2^{(3)} + \frac{M_1K_2 + M_2(K_1+K_2)}{M_1M_2}\ddot{z}_2 + \frac{B_1K_2}{M_1M_2}\dot{z}_2 + \frac{K_1K_2}{M_1M_2}z_2 = \frac{K_2}{M_1M_2}f$$

Input/Output Models vs State-Space Models

- **State Space Models:**
  - consider the internal behavior of a system
  - can easily incorporate complicated output variables
  - have significant computation advantage for computer simulation
  - can represent multi-input multi-output (MIMO) systems and nonlinear systems
- **Input/Output Models:**
  - are conceptually simple
  - are easily converted to frequency domain transfer functions that are more intuitive to practicing engineers
  - are difficult to solve in the time domain (solution: Laplace transformation)

Conversion between different models

Converting From	Converting to	Matlab function
Transfer Function	Zero-pole-gain	[z,p,k]=tf2zp(num,den)
Transfer Function	State Space	[A,B,C,D]=tf2ss(num,den)
Zero-pole-gain	Transfer Function	[num,den]=zp2tf(z,p,k)
Zero-pole-gain	State Space	[A,B,C,D]=zp2ss(z,p,k)
State Space	Transfer Function	[num,den]=ss2tf(A,B,C,D)
State Space	Zero-pole-gain	[z,p,k]=ss2zp(A,B,C,D)

State Space Equations

Dynamic Equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

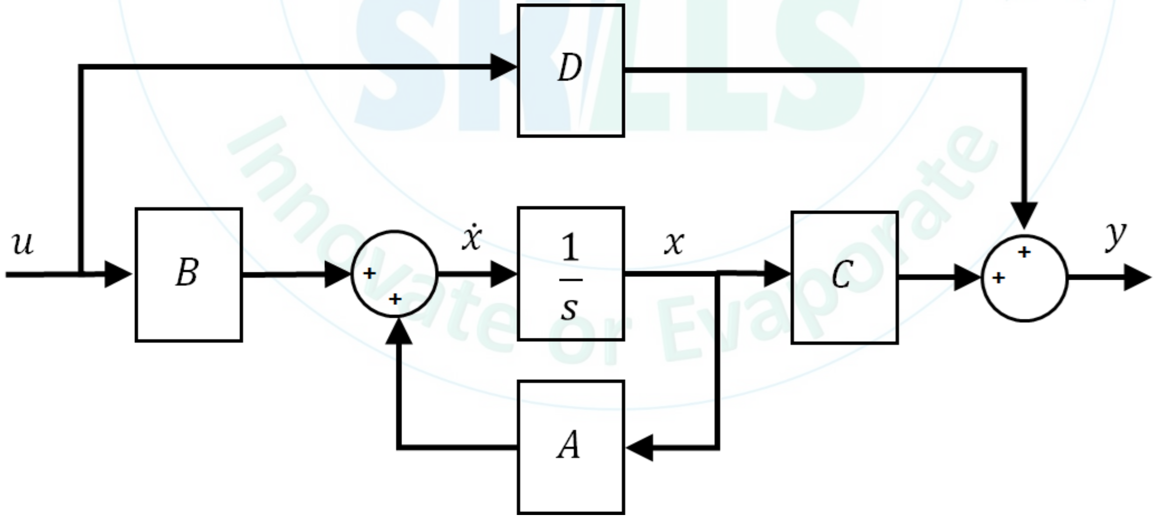
Observation Equations

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

x is the state vector of the system, nx1;
n is the order of the system;
u is the input vector, mx1;
y is the output vector, px1;
A is the system (or coefficient) matrix, nxn;
B is the input (or driving) matrix, nxm;
C is the observation matrix, pxn;
D is the feedforward matrix, pxm.

MATLAB function:

```
SYS = SS(A,B,C,D
SYS = SS(A,B,C,D,T)
SYS = SS
SYS = SS(D)
SYS =
SS(A,B,C,D,LTISYS)
SYS = SS(SYS)
```



State Space Model to Transfer Function Model

MATLAB function:

```
[NUM,DEN] =
SS2TF(A,B,C,D,iu)
```

$s\mathbf{x}(s) - \mathbf{x}(0) = A\,\mathbf{x}(s) + B\,\mathbf{u}(s)$

Solving for $\mathbf{x}(s)$,

$\mathbf{x}(s) = \text{inv}(sI-A)\,\mathbf{x}(0) + \text{inv}(sI-A)\,B\,\mathbf{u}(s).$

So,

$$\mathbf{y}(s) = C\,\mathbf{x}(s) + D\,\mathbf{u}(s)$$

$$= C\,\text{inv}(sI-A)B\,\mathbf{u}(s) + D\,\mathbf{u}(s) \quad \text{for } \mathbf{x}(0) = \mathbf{0}.$$

$$=[C\,\text{inv}(sI-A)B + D]\mathbf{u}(s)$$

$$=\{[C\,\text{adj}(sI-A)B + D\,\det(sI-A)] / \det(sI-A)\}\,\mathbf{u}(s)$$

$\det(sI-A) = 0$ is the *characteristic equation* of the system and
$[C\,\text{adj}(sI-A)\,B + D\,\det(sI-A)]$ is the pxm *matrix of system zeros*

State Space Model to Transfer Function Model

State equations as functions of time

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

In the s-domain

$$sX - X_0 = AX + BU$$

$$X(sI - A) = BU + X_0$$

$$X = (sI - A)^{-1}BU + (sI - A)^{-1}X_0$$

$$Y = CX + DU$$

$$Y = C((sI - A)^{-1}BU + (sI - A)^{-1}X_0) + DU$$

$$Y = (C(sI - A)^{-1}B + D)U + C(sI - A)^{-1}X_0$$

Assuming the system starts at rest,

$$Y = (C(sI - A)^{-1}B + D)U$$

$$\frac{Y}{U} = (C(sI - A)^{-1}B + D) \qquad \text{(the transfer function)}$$

State Space Modeling

Example:

State-space representations are, in general, not unique. One system can be represented in several possible ways. For example, consider the following systems:

a.
$$\dot{x} = -5x + 3u$$
$$y = 7x$$

b.
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 7 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

c.
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 7 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Show that these systems will result in the same transfer function. We will explore this phenomenon

a.
$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = 7(s+5)^{-1}3 = \frac{21}{(s+5)}$$

Pole at s=-5

1 state variable & one pole

b.
$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 7 & 0 \end{bmatrix} \begin{bmatrix} s+5 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+5} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{7}{s+5} & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{21}{s+5}$$

c)
$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 7 & 3 \end{bmatrix} \begin{bmatrix} s+5 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{s+5} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{s+5} \\ 0 \end{bmatrix} = \frac{21}{s+5}$$

2 state variables & one pole

but, two eigenvalues

```
>> A=[-5   0;0    -1];
>> eig(A)
ans =  -5     -1
```

Controllability and Observability

The Controllability Matrix

An nth-order plant whose state equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

is completely controllable if the matrix

$$\mathbf{C_M} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}]$$

is of rank n, where $\mathbf{C_M}$ is called the *controllability* matrix.

Controllability and Observability

Example:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$\mathbf{C}_M = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{bmatrix}$$

The rank of C_M equals the number of linearly independent rows or columns. The rank can be found by finding the highest-order square submatrix that is nonsingular.
The determinant of $\mathbf{C_M}$ = -1. Since the determinant is not zero, the 3X3 matrix is nonsingular, and the rank of $\mathbf{C_M}$ is 3.

```
A=[-1 1 0;0 -1 0;0 0 -2]      % Define compensated A matrix.
B=[0;1;1]                     % Define compensated B matrix.
Cm=ctrb(A,B)                  % Calculate controllability matrix.
Rank=rank(Cm)                 % Find rank of controllability matrix.
```

The Observability Matrix

An nth-order plant whose state and output equations are, respectively,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

is completely observable[6] if the matrix

$$\mathbf{O_M} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

is of rank n, where $\mathbf{O_M}$ is called the *observability matrix*.

Controllability and Observability

Example: The state and output equations for the system are

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} 0 & 1 \\ -5 & -21/4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$y = \mathbf{C}\mathbf{x} = [5 \quad 4]\mathbf{x}$$

The observability matrix, $\mathbf{O_M}$, for this system is

$$\mathbf{O_M} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -20 & -16 \end{bmatrix}$$

The rank of **O_M** equals the number of linearly independent rows or columns. The rank can be found by finding the highest-order square submatrix that is nonsingular. The determinant of **O_M** = 0. Since the determinant is zero, the 2 X 2 matrix is singular, and the rank of **C_M** is equal to 1.

```
A=[0 1;-5 -21/4]      % Define compensated A matrix.
C=[5 4]               % Define compensated C matrix.
Om=obsv(A,C)          % Calculate observability matrix.
Rank=rank(Om)         % Find rank of observability matrix.
```

Controllability and Observability

Example:

The state and output equations for the system are

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \mathbf{C}\mathbf{x} = [0 \quad 5 \quad 1]\mathbf{x}$$

$$\mathbf{O_M} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 1 \\ -4 & -3 & 3 \\ -12 & -13 & -9 \end{bmatrix}$$

The rank of **OM** equals the number of linearly independent rows or columns. The rank can be found by finding the highest-order square submatrix that is nonsingular. The determinant of **O_M** = -344. Since the determinant is not zero, the 3X3 matrix is nonsingular, and the rank of **C_M** is of full rank equal to 3.

```
A=[0 1 0;0 0 1;-4 -3 -2]     % Define compensated A matrix.
C=[0 5 1]                    % Define compensated C matrix.
Om=obsv(A,C)                 % Calculate observability matrix.
Rank=rank(Om)                % Find rank of observability matrix.
```

Controllability and Observability

$$A = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} ------(1)$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} ------(2)$$

$$C = [1 \quad 2 \quad 3] ------(3)$$

Since A is a 3x3 matrix, so n=3.

Controllability: The necessary and sufficient condition for controllability is

$$Q_C = [B \quad AB \quad ------ \quad A^{n-1}B]$$

$$Q_C = [B \quad AB \quad A^2B] ------(4)$$

From equation (2),

$$B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} ------(5)$$

$$A \cdot B = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} -4 \\ 4 \\ 10 \end{bmatrix} ------(6)$$

$$A^2B = A \cdot (AB) = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \\ 10 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} -26 \\ 16 \\ 36 \end{bmatrix} ------(7)$$

Put equation (5), (6), (7) in equation (4),

$$Q_C = \begin{bmatrix} 0 & -4 & -26 \\ 1 & 4 & 16 \\ 2 & 10 & 36 \end{bmatrix}$$

Now, find the determinant of Q_C

$$|Q_C| = \begin{vmatrix} 0 & -4 & -26 \\ 1 & 4 & 16 \\ 2 & 10 & 36 \end{vmatrix}$$

$$|Q_C| = -36$$

$$\therefore |Q_C| \neq 0$$

Since the determinant of Q_C is non-zero, therefore of $Q_C = n = 3$. Hence, the system is completely controllable.

Controllability and Observability

Observability:

Given,

$$B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$C' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 1 & 3 \\ 6 & 0 & 2 \\ -5 & 2 & 4 \end{bmatrix}$$

The necessary and sufficient condition for observability.

$$Q_C = \begin{bmatrix} C^T & A^T C^T & ---- & (A^T)^{n-1} C^T \end{bmatrix}$$

For n=3,

$$Q_C = \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T \end{bmatrix} ------ (8)$$

$$C^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} ------ (9)$$

$$A^T C^T = \begin{bmatrix} 0 & 1 & 3 \\ 6 & 0 & 2 \\ -5 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 11 \\ 12 \\ 11 \end{bmatrix} ------ (10)$$

$$(A^T)^2 C^T = A^T \cdot (A^T C^T) = \begin{bmatrix} 0 & 1 & 3 \\ 6 & 0 & 2 \\ -5 & 2 & 4 \end{bmatrix} \begin{bmatrix} 11 \\ 12 \\ 11 \end{bmatrix}$$

$$(A^T)^2 C^T = \begin{bmatrix} 45 \\ 88 \\ 13 \end{bmatrix} ------ (11)$$

Put equations (9), (10), (11) in equation (8),

$$Q_C = \begin{bmatrix} 1 & 11 & 45 \\ 2 & 12 & 88 \\ 3 & 11 & 13 \end{bmatrix}$$

Now, find the determinant of Q_C

$$|Q_C| = \begin{vmatrix} 1 & 11 & 45 \\ 2 & 12 & 88 \\ 3 & 11 & 13 \end{vmatrix}$$

$$|Q_C| = 1176$$

Since, $|Q_C| \neq 0$, the rank of Q_C is n=3.

$\therefore$ The system is completely observable.

END

Controller Canonical Form

Consider the transfer function

$$\frac{y(s)}{u(s)} = \sum_{i=0}^{n-1} b_i s^i \Big/ \sum_{j=0}^{n} a_j s^{n-j}, a_0 = 1.$$

Define

$$y = \sum_i b_i x_{c_i} \quad \text{, and} \quad \dot{x}_{c_i} = x_{c_{i-1}}, i = 2, \ldots n$$

Structure of the state space model

$$\dot{\mathbf{x}}_\mathbf{c} = \mathbf{A}_\mathbf{c}\mathbf{x}_\mathbf{c} + \mathbf{B}_\mathbf{c}\mathbf{u}$$

$$\mathbf{A_c} = \begin{bmatrix} -a_1 - a_2 - a_3 \ldots\ldots\ldots\ldots -a_{n-1} - a_n \\ \mathbf{I}_{(n-1)x(n-1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{0}_{(n-1)x1} \end{bmatrix}_{(nxn)}$$

$$\mathbf{B_c} = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}_{(nx1)} , \mathbf{C_c} = [b_1 \quad \cdots \quad b_n] \text{ , } \mathbf{D_c} = \mathbf{0}.$$

Controller Canonical Form

Controller canonical form is particularly useful:

- The coefficients of the numerator and denominator
- Polynomials of the transfer function appear *directly* in the state variable model.
- All other elements are either *0 or 1*.
- The *state variable model can be written by inspection*, and vice versa. This is used in MATLAB to compute the state space model from the transfer function model with the function **tf2ss**.
- *It replicates itself if state variable feedback is used.*

Controller Canonical Form

Consider a state variable feedback control law.

$$u(t) = -\mathbf{K}\mathbf{x_c}(t), \mathbf{K} = \left[k_1 \ldots\ldots k_n\right]$$

$$\dot{\mathbf{x}}_\mathbf{c} = \mathbf{A_c}\mathbf{x_c} - \mathbf{B_c}\mathbf{K}\mathbf{x_c} = \left[\mathbf{A_c} - \mathbf{B_c}\mathbf{K}\right]\mathbf{x_c}$$

$$\left[\mathbf{A_c} - \mathbf{B}_c\mathbf{K}\right] = \begin{bmatrix} -(a_1 + k_1) \ldots\ldots\ldots\ldots -(a_n + k_n) \\ \mathbf{I}_{(n-1)x(n-1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{0}_{(n-1)x1} \end{bmatrix}$$

Thus, the closed-loop characteristic equation is:

$$s^n + (a_1 + k_1)s^{n-1} + (a_2 + k_2)s^{n-2} + \ldots\ldots\ldots + (a_n + k_n) = 0$$

The feedback coefficients determine the closed-loop poles.

State Space Representations

Example 1.1: Conversion from Transfer-Function Model to State-Space Model

$$\frac{Y(s)}{U(s)} = \frac{s}{s^3+14s^2+56s+160}$$

MATLAB code:

```
n=[1 0]; d=[1 14 56 160];

[Ac,Bc,Cc,Dc]=tf2ss(n,d)  %Controller canonical form

system=tf(n,d)

[a,b,c,d]=ssdata(system)   % Not controller canonical form
```

State Space Representations

Example 1:

$$\frac{s}{s^3 + 14 s^2 + 56 s + 160}$$

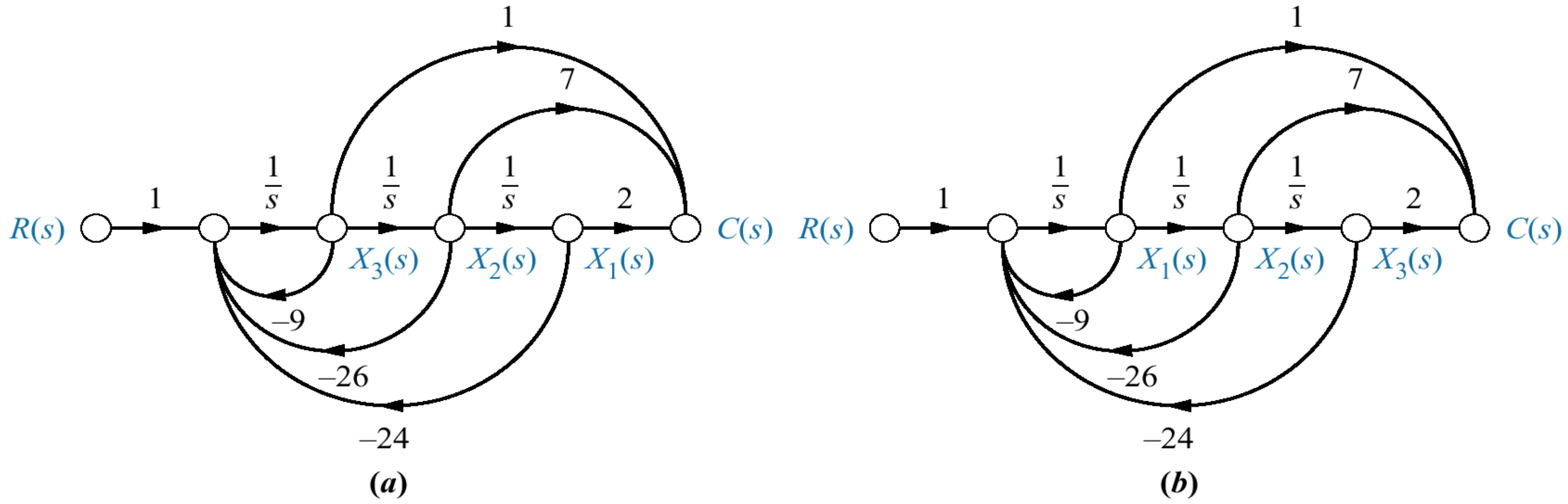
```
Ac =
   -14   -56  -160
     1     0     0
     0     1     0
Bc =
     1
     0
     0
Cc =
     0     1     0
Dc =
     0
```

```
a =
   -14    -7    -5
     8     0     0
     0     4     0
b =
    0.2500
         0
         0
c =
         0    0.5000         0
d =
     0
```

Similarity between phase-variable & controller canonical forms

Signal-flow graphs for obtaining forms for

$$G(s) = C(s)/R(s) = (s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$$



a. phase-variable form; **b.** controller canonical form

Modal Canonical Form

MATLAB function:

```
[R,P,K] = RESIDUE(B,A)
[B,A] = RESIDUE(R,P,K)
```

For non-repeated roots

$$y(s)/u(s) = \sum_{i=1}^{n} r_i/(s-p_i) \quad \text{for} \quad p_i \neq p_j \quad i \neq j$$

Let $x_{m_i}(s)/u(s) = r_i/(s-p_i)$

So,

$$\dot{x}_{m_i}(t) = p_i x_{m_i}(t) + r_i u(t)$$

$$\mathrm{y(t)} = \sum_{i=1}^{\mathrm{n}} x_{m_i}(t) = [1\ldots\ldots1]\mathbf{x}$$

Modal Canonical Form - Vector-Matrix Format

The state variables in this format are *uncoupled*. they are called the *modes*. In vector-matrix notation,

$$\dot{\mathbf{x}}_{\mathbf{m}}(t) = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & p_n \end{bmatrix} \mathbf{x}_{\mathbf{m}}(t) + \begin{bmatrix} r_1 \\ \vdots \\ \vdots \\ r_n \end{bmatrix} u(t)$$

and

$$y = [1 \ldots 1]\mathbf{x}_{\mathbf{m}}.$$

Definition of Controllability

- If an input can be found that takes *every* state variable from an *arbitrary initial state* to a *desired final state* in a *finite amount of time*, the system is said to be *controllable*;
- otherwise, the system is *uncontrollable*.

- The *controllability* of a system in *modal canonical form* can be determined by *inspection*.

- *A system is controllable iff all of its modes can be affected by the control (input) variable(s).*

Consider a scalar system

$$\dot{x} = px + ru$$

$$x(T) = e^{p(T-t_0)}x(t_0) + \int_{t_0}^{T} e^{p(T-\tau)}ru(\tau)d\tau$$

For u = constant = U,

$$U = \frac{x(T) - e^{p(T-t_0)}}{r\int_{t_0}^{T} e^{p(T-\tau)}d\tau}$$

Thus, the scalar system is *controllable iff r is not zero* and the controllability of each mode can be assessed by inspection of the modal canonical form.

Transforming State Equations

$\mathbf{x} = \mathbf{T}\mathbf{z}$

then $\dot{\mathbf{x}} = \mathbf{T}\dot{\mathbf{z}} = \mathbf{A}\mathbf{T}\mathbf{z} + \mathbf{B}\mathbf{u}$

or $\dot{\mathbf{z}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z} + \mathbf{T}^{-1}\mathbf{B}\mathbf{u} = \mathbf{F}\mathbf{z} + \mathbf{G}\mathbf{u}$

and $\mathbf{y} = \mathbf{C}\mathbf{T}\mathbf{z} + \mathbf{D}\mathbf{u} = \mathbf{H}\mathbf{z} + \mathbf{J}\mathbf{u}$.

So that the state space models are related by the matrix transformations:

$\mathbf{F} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$

$\mathbf{G} = \mathbf{T}^{-1}\mathbf{B}$

$\mathbf{H} = \mathbf{C}\mathbf{T}$

$\mathbf{J} = \mathbf{D}$.

Transforming to Control Canonical Form

Suppose we wanted to transform an arbitrary state variable description (**A,B,C,D**) to the control canonical form.
Is it possible?

From the general transformation equations,

$$\mathbf{A}_c\mathbf{T} = \mathbf{T}\ \ \mathbf{A}.$$

Considering the three-dimensional case and letting:

$$\mathbf{T} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \mathbf{t}_3 \end{bmatrix}^T,$$

Transforming to Control Canonical Form

$$\begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}_1^T \\ \mathbf{t}_2^T \\ \mathbf{t}_3^T \end{bmatrix} = \begin{bmatrix} \mathbf{t}_1^T \\ \mathbf{t}_2^T \\ \mathbf{t}_3^T \end{bmatrix} \mathbf{A}$$

From the last row, $\mathbf{t}_2^T = \mathbf{t}_3^T \mathbf{A}$.

From the second row, $\mathbf{t}_1^T = \mathbf{t}_2^T \mathbf{A}$.

Also, since $\mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{TB} = \begin{bmatrix} \mathbf{t}_1^T \\ \mathbf{t}_2^T \\ \mathbf{t}_3^T \end{bmatrix} \mathbf{B}$,

$1 = \mathbf{t}_1^T \mathbf{B} = \mathbf{t}_2^T \mathbf{AB} = \mathbf{t}_3^T \mathbf{A}^2 \mathbf{B}$

$0 = \mathbf{t}_2^T \mathbf{B} = \mathbf{t}_3^T \mathbf{A} \ \mathbf{B}$

$0 = \mathbf{t}_3^T \mathbf{B}$

or

$\mathbf{t}_3^T [\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}] = [0 \quad 0 \quad 1]$.

Transforming to Control Canonical Form

Solving for $\mathbf{t}_3^T$,

$$\mathbf{t}_3^T = [0 \quad 0 \quad 1][\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}]^{-1}$$

From which we can solve for $\mathbf{t}_2^T$ and $\mathbf{t}_1^T$ from the equation derived above if, and only if, $[\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}]^{-1}$ exists!

General Case: In general, we define the controllability matrix, **C**, as:

$$\boldsymbol{C} = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$$

$$\mathbf{t}_n^T = \begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}\boldsymbol{C}^{-1}$$

$$\mathbf{T} = \begin{bmatrix}\mathbf{t}_n^T\mathbf{A}^{n-1} \\ \mathbf{t}_n^T\mathbf{A}^{n-2} \\ \vdots \\ \mathbf{t}_n^T\end{bmatrix}$$

MATLAB function:

```
CO = CTRB(A,B)
CO = CTRB(SYS)
```

Controllability - an alternate viewpoint

- We have just seen that an *arbitrary* state space representation can be transformed to the *control canonical* form *iff the controllability matrix is nonsingular.*

- Once in controller canonical form the closed-loop poles can be arbitrarily placed using state variable feedback.

- Thus, *a system is controllable iff its closed-loop poles can be arbitrarily selected using state variable feedback.*

- All *viewpoints of controllability are compatible*.

Forming the Controllability Matrix

Example

For the canonical state-space model defined in Example 2

CO = ctrb(Ac,Bc)

CO =

```
    1   -14   140
    0     1   -14
    0     0     1
```

Now we can check its determinant to make sure its inverse exists:

det(CO)

ans = 1

Therefore, the system is controllable.

An Uncontrollable System

Example 3: Let **a=[1 1;0 1]; b=[1;0]; CO=ctrb(a,b); det(CO)**

ans = 0

Since the controllability matrix is singular the system is uncontrollable. Note the transfer function model has pole-zero cancellation at s = -1. This is symptomatic of an uncontrollable mode.

[zeros,poles]=ss2zp(a,b,[1 1],0)

zeros =
  1.0000
poles =
  1
  1

Transforming to Modal Canonical Form

$$\dot{\mathbf{z}} = \mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{x} + \mathbf{V}\mathbf{B}\mathbf{u} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}\ \mathbf{z} + \mathbf{V}^{-1}\mathbf{B}\mathbf{u} = \mathbf{A}_{\mathbf{m}}\mathbf{z} + \mathbf{B}_{\mathbf{m}}\mathbf{u}.$$

So, $\mathbf{A}_{\mathbf{m}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ or $\mathbf{V}\mathbf{A}_{\mathbf{m}} = \mathbf{A}\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_n \end{bmatrix}.$

Thus, $\mathbf{A}\mathbf{v}_i = p_i\mathbf{v}_i$ for $i = 1\cdots n$ **Eigenvalue Problem**

where

$\{p_i\}$ are the eigenvalues of $\mathbf{A}$, i.e., $\det(\mathbf{A} - \mathrm{p_i}\mathbf{I}) = 0$; and

$\{\mathbf{v}_i\}$ are the eigenvectors of $\mathbf{A}$.

MATLAB function:

```
E = EIG(X)
[V,D] = EIG(X)
```