MENG366

Transient and Steady State Response of First, Second, and Higher Order Systems

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Forced Responses of LTI Systems

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_m u^{(m)} + \cdots + b_1\dot{u} + b_0 u$$

- **Superposition Principle**

$u_1(t)$ → $y_1(t)$

$u_2(t)$ → $y_2(t)$

$u(t) = k_1 u_1(t) + k_2 u_2(t)$ → $y(t) = k_1 y_1(t) + k_2 y_2(t)$

complicated input $= \sum$ simple inputs

$\sum$ forced responses of simple inputs = forced response of complicated input

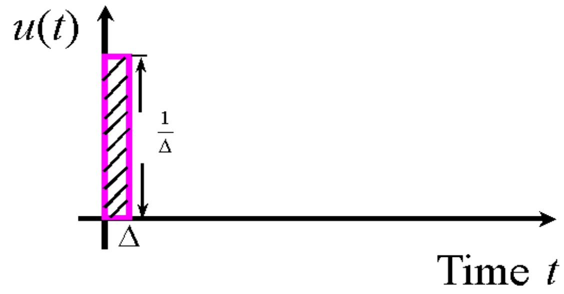
The forced response of a linear system to a complicated input can be obtained by studying how the system responds to simple inputs, such as unit impulse input, unit step input, and sinusoidal inputs with different input frequencies.

Typical Forced Responses

- **Unit Impulse Response**
  - Forced response to unit impulse input

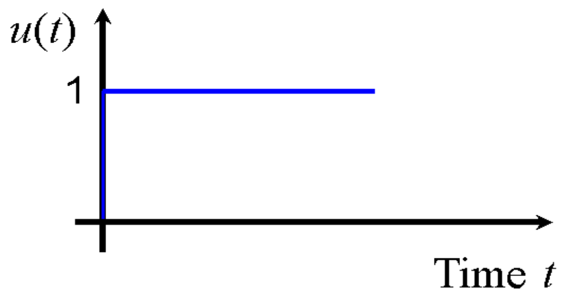
$$u(t)=\delta(t)=\begin{cases}+\infty, & t=0\\ 0, & t\neq 0\end{cases},\quad \int_{-\infty}^{+\infty}\delta(t)\,dt=1\quad Y(s)=G(s)U(s)=G(s)$$

If system is stable, *SS* is zero.



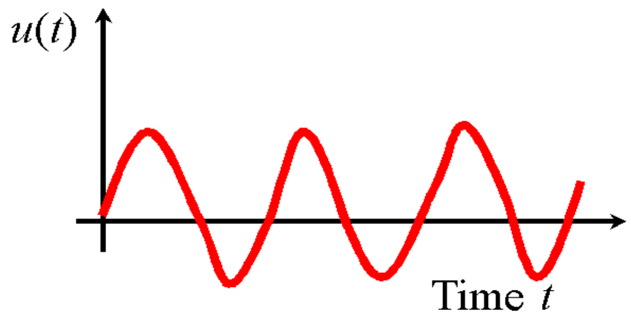
- **Unit Step Response**
  - Forced response to unit step input (u (t) = 1)

$$Y(s)=G(s)U(s)=\underbrace{\frac{1}{s}}_{U(s)}G(s)\qquad y(\infty)=\lim_{s\to 0}sY(s)=\lim_{s\to 0}s\frac{1}{s}G(s)=G(0)$$



- **Sinusoidal Response**
  - Forced response to sinusoidal inputs at different input frequencies
  - The *steady state response* of sinusoidal response is call the *Frequency Response*.

$$Y(s)=G(s)U(s)=G(s)\underbrace{\frac{\omega}{s^2+\omega^2}}_{U(s)}\qquad sY(s)=\frac{s\omega}{s^2+\omega^2}G(s)$$



Forced Response of 1st Order Systems

- **Standard Form of *Stable* 1st Order System**

$$\dot{y} + ay = b\,u \quad \Rightarrow \quad \boxed{\tau\dot{y} + y = K\,u}$$

$$\frac{1}{a}\dot{y} + y = \frac{b}{a}\,u$$

where

τ : Time Constant $\qquad \tau = \dfrac{1}{a}$

K : Static (Steady State, DC) Gain $\qquad K = \dfrac{b}{a}$

- **TF and Poles/Zeros**

$$G(s) = \frac{K}{\tau s + 1} \qquad \text{pole: } p = -\frac{1}{\tau} < 0 \qquad \text{zero: No zero}$$

Stable system

Forced Response of 1st Order Systems

Impulse response:

$$C(s) = \frac{1}{\tau s + 1} R(s)$$

$$r(t) = A\delta(t)$$

$$R(s) = A$$ Impulse input

If A=1 it is called unit impulse

Some impulse input examples:

- Impulse voltage (electronic flash gun)
- Impulse force (hammer hit)
- Impulse torque (pneumatic nut remover)
- Impulse pressure (explosion)
- Impulse displacement
- Impulse temperature (thermal shock)

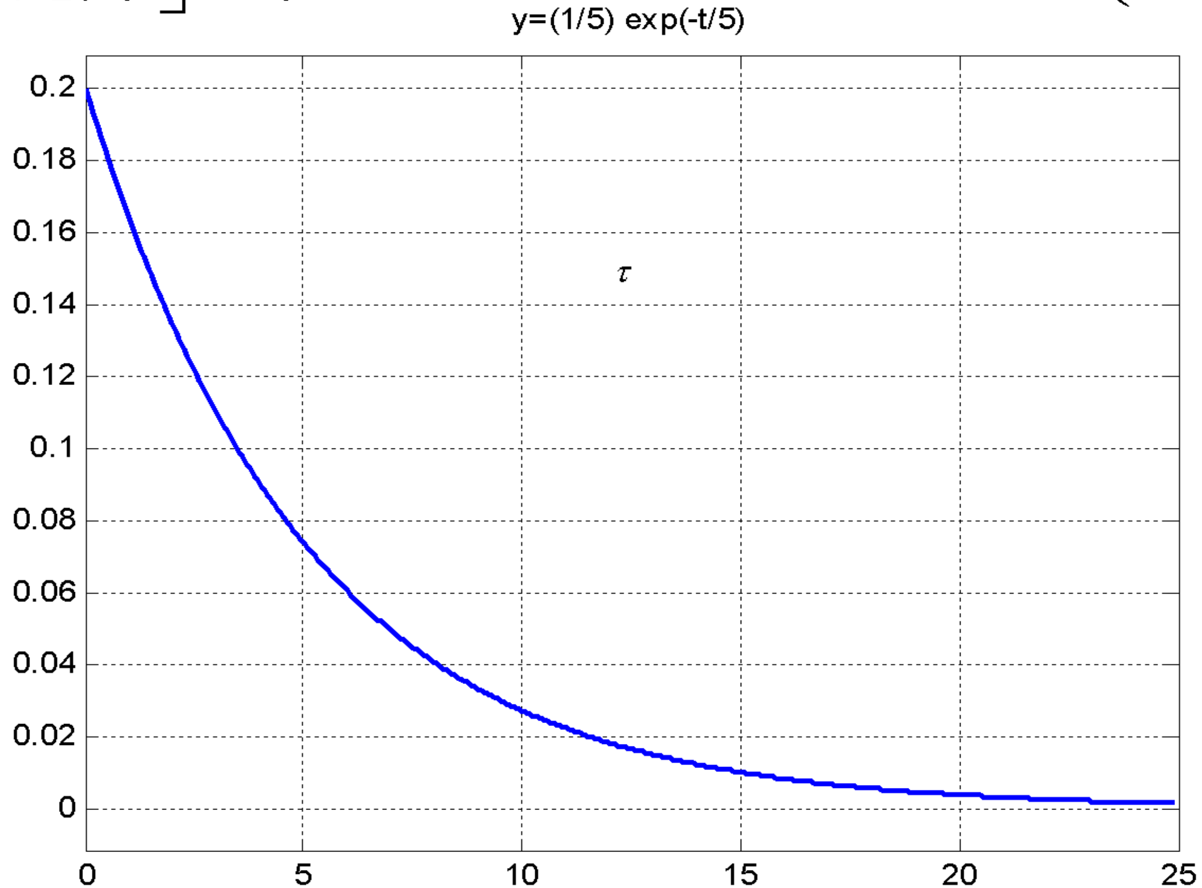
$$C(s) = \frac{A}{\tau s + 1}$$

Forced Response of 1st Order Systems

Take inverse Laplace Transform:

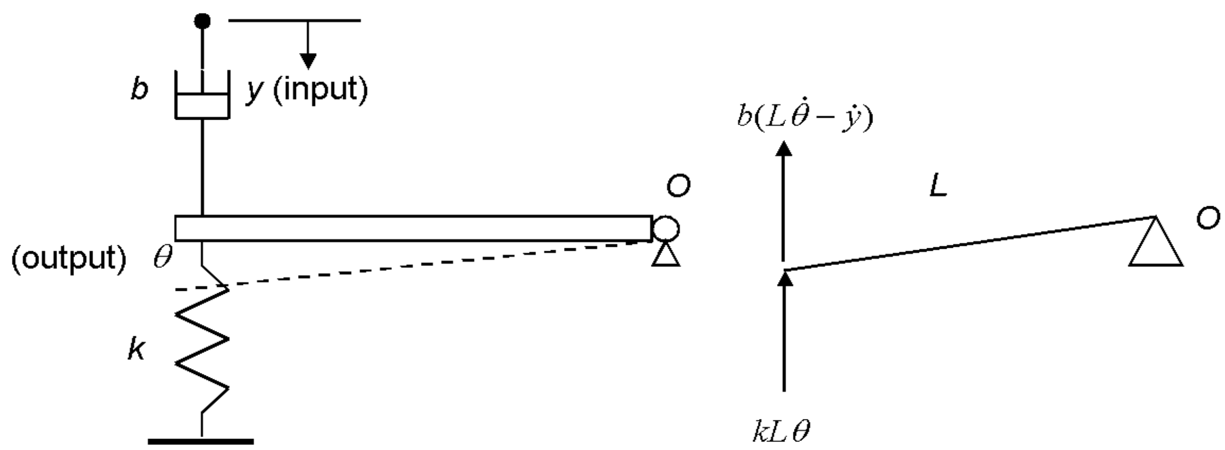
$$c(t) = L^{-1}\left[\frac{A/\tau}{s+1/\tau}\right] = \frac{A}{\tau}e^{-\frac{t}{\tau}}, \quad t \geq 0$$

$$y(t) = K\left(1 - e^{-\frac{1}{\tau}}\right)$$



Unit impulse response, time constant , τ =5 s.

Example:



Obtain unit step response of the mechanical system. Ignore mass of the bar (J=0).

$$\sum M_o = 0$$

$$L\sin\theta \approx L\theta$$

$$y > L\theta$$

$$b(\dot{y} - L\dot{\theta})L - kL\theta L = 0$$

$$bL\dot{\theta} + kL\theta = b\dot{y}$$

Take L.T.

$$(bLs + kL)\theta(s) = bsY(s)$$

$$\frac{\theta(s)}{Y(s)} = \frac{bs}{bLs + kL}$$

$$Y(s) = \frac{1}{s}, \quad \text{unit step input}$$

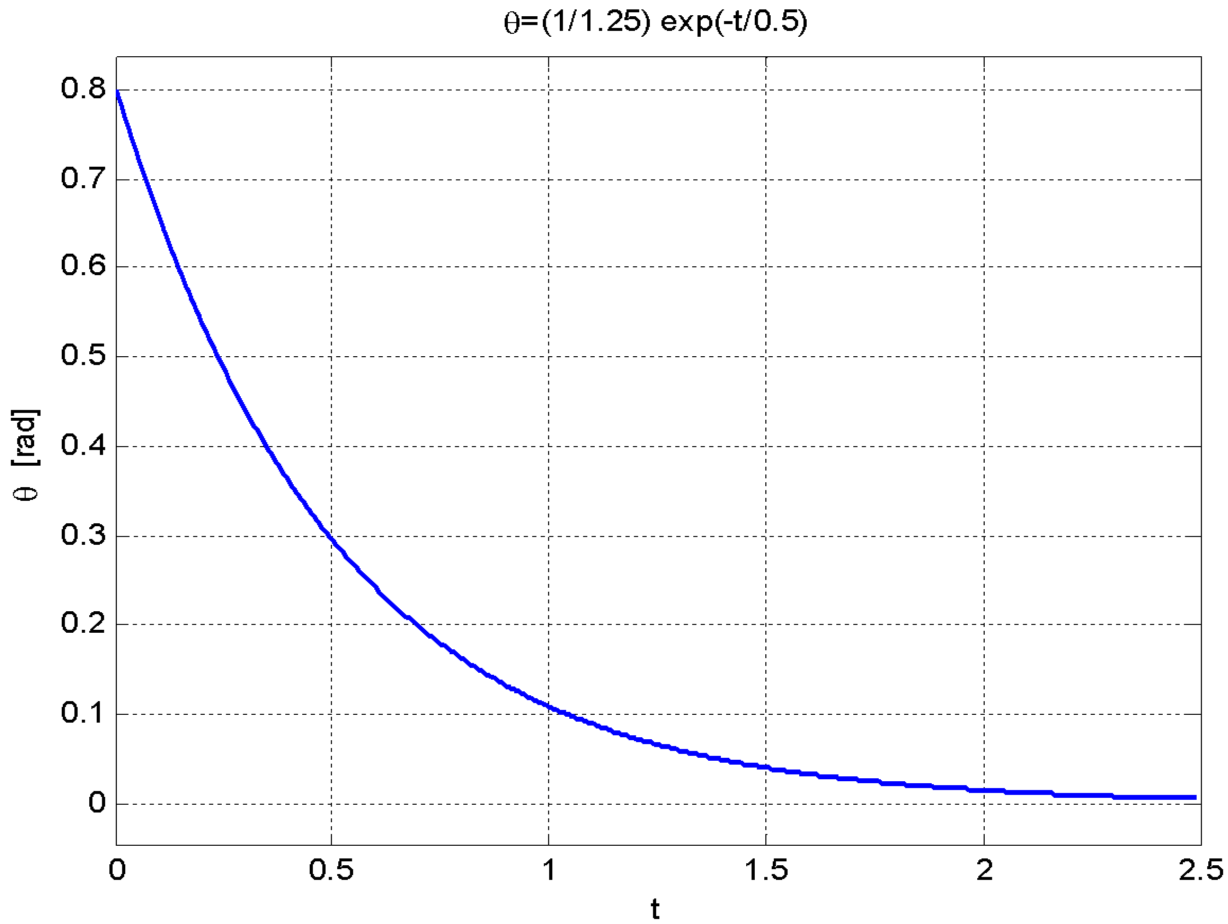
$$\theta(\mathrm{s}) = \frac{bs}{bLs + kL} \cdot \frac{1}{s} = \frac{1/L}{s + \frac{k}{b}}$$

$$\theta(t) = \frac{1}{L} e^{-\frac{kt}{b}}$$

Example (cont.):

Let:

L=1.25 m
b=10 Ns/m
k=20 N/m
T=b/k=10/20=0.5 s
1/L=1/1.25=0.8 rad

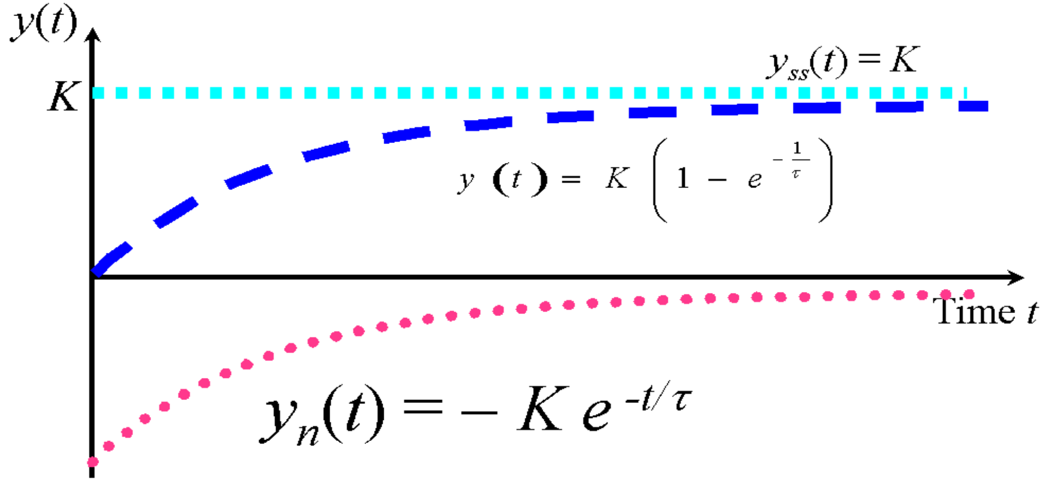


Forced Response of 1st Order Systems

- **Unit Step Response:**
  - (u=1 and zero ICs)

$$Y(s) = G(s)U(s) = \frac{K}{(\tau s+1)s} = K\left[\frac{1}{s} - \frac{\tau}{\tau s+1}\right] = K\left[\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right]$$

$$y(t) = K\left(1 - e^{-\frac{1}{\tau}}\right)$$



Normalized Unit Step Response

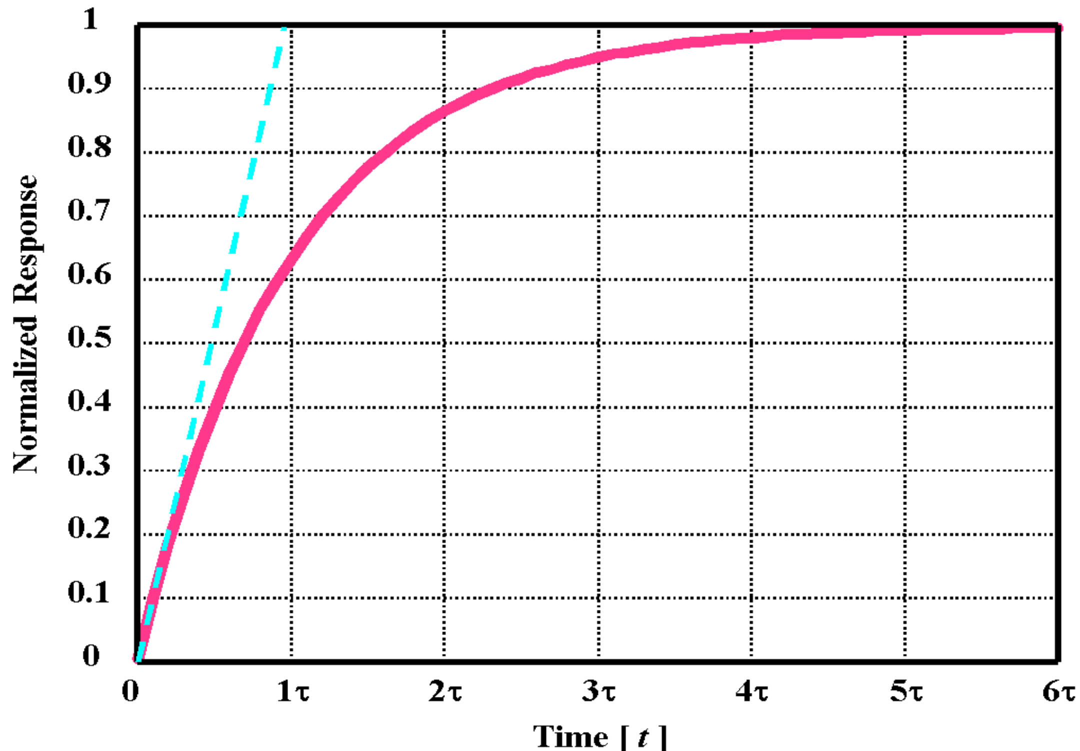
Normalized Unit Step Response (u = 1 & zero ICs)

$$\tau \dot{y} + y = K u$$

$$\Rightarrow \quad y(t) = K(1 - e^{-t/\tau})$$

Normalized
(such that as $t \to \infty$, $y_n \to 1$):

$$\Rightarrow \quad y_n(t) = \frac{y(t)}{K} = 1 - e^{-t/\tau}$$



Time t	τ	2τ	3τ	4τ	5τ
$(1 - e^{-t/\tau})$	0.6321	0.8647	0.9502	0.9817	0.9933

Unit Step Response of Stable 1st Order System

Effect of Time Constant τ :

$$\tau \dot{y} + y = K u$$

$$\Rightarrow \quad y(t) = K(1 - e^{-t/\tau})$$

Normalized:

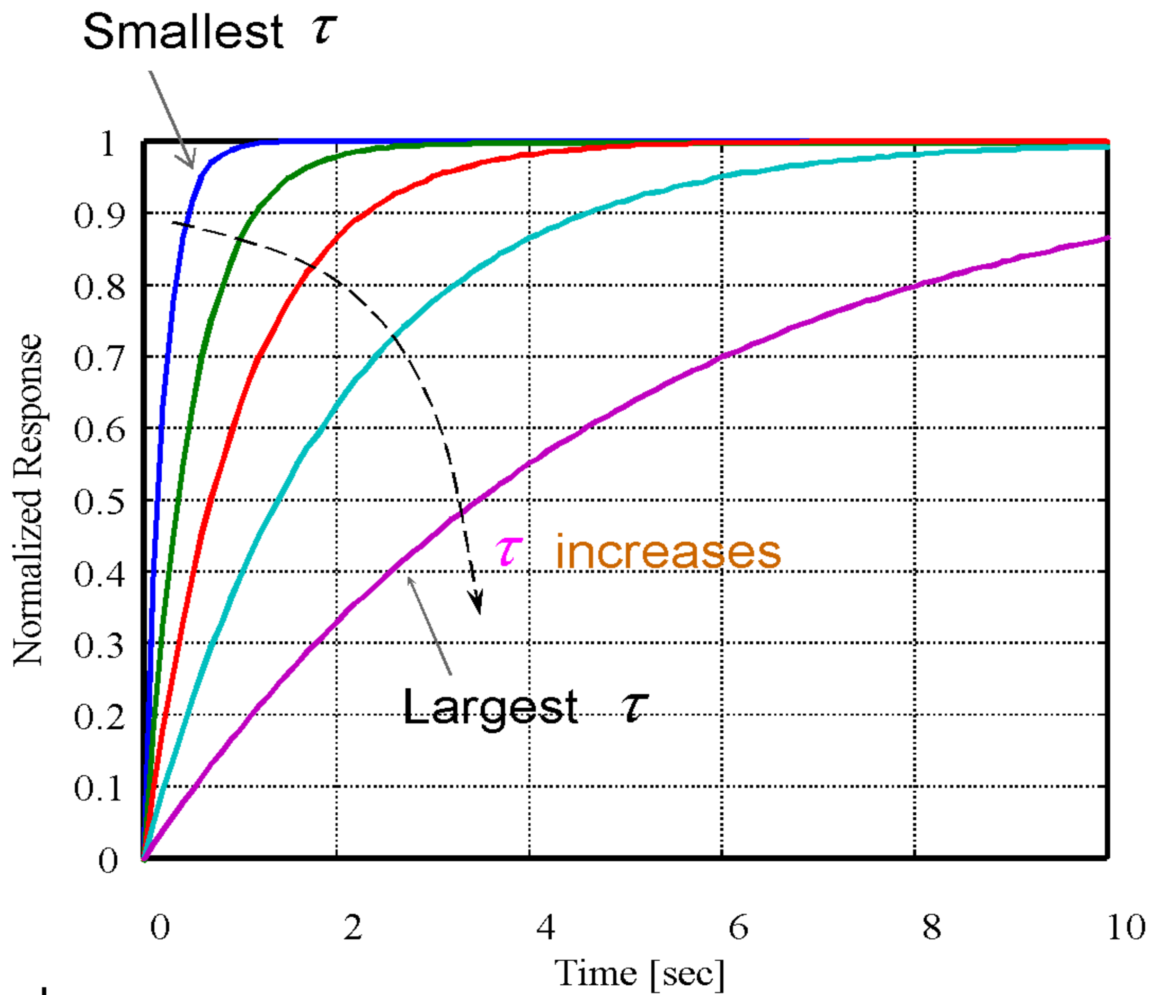
$$\Rightarrow \quad y_n(t) = \frac{y(t)}{K} = (1 - e^{-t/\tau})$$

Initial Slope:

$$\Rightarrow \quad \frac{d}{dt} y_n(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}$$

$$\Rightarrow \quad \frac{d}{dt} y_n(0) = \frac{1}{\tau}$$

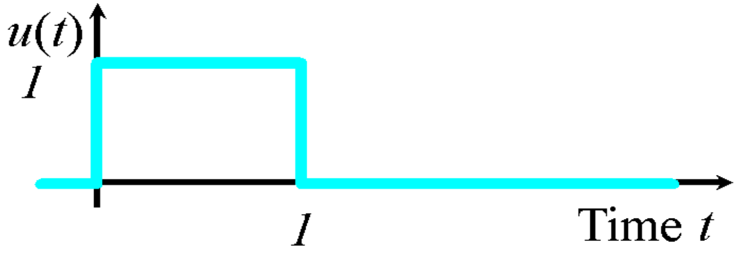
Q: What is your conclusion ?



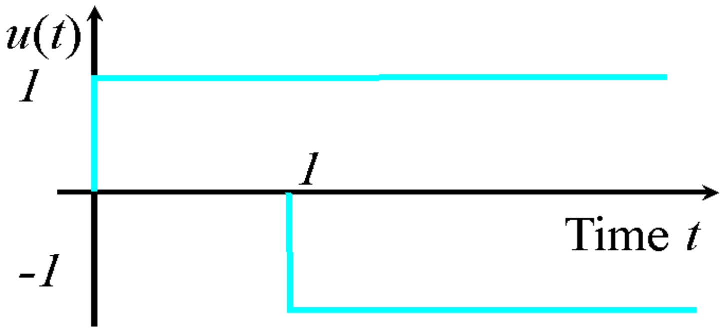
The smaller τ is,
the steeper the initial slope is, and
the faster the response approaches the steady state.

Forced Responses of Stable 1st Order System

Q: *How would you calculate the forced response of a 1st order system to a <u>unit pulse</u> (not unit impulse)?*



Q (Hint: superposition principle ?!)



$$u(t) = u_s(t) - u_s(t-1)$$

$$y(t) = y_s(t) - y_s(t-1)$$

Q: *How would you calculate the unit impulse response of a 1st order system?*

$$\delta(t) = \frac{d}{dt} u_s(t)$$

$$y(t) = \frac{d}{dt} y_s(t)$$

Q: *How would you calculate the sinusoidal response of a 1st order system?*

Standard Form of 2nd Order Systems

- **I/O Model**

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{u} + b_0 u$$

- **TF and Pole/Zeros**

$$G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \qquad \text{pole: } p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

$$\omega_n = \sqrt{a_0} \qquad \text{zero: } z = -\frac{b_0}{b_1}$$

- **Stability Condition**

$$a_1 > 0, \quad a_0 > 0$$

- **Standard Form of *Stable* 2nd Order Systems without Zeros**

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_0 u \quad \Rightarrow \quad \ddot{y} + \underbrace{2\zeta\omega_n}_{a_1} \dot{y} + \underbrace{\omega_n^2}_{a_0} y = \underbrace{K\omega_n^2}_{b_0} u$$

where

ω_n : **Natural Frequency [rad/s]**

ζ : **Damping Ratio** $\quad \zeta = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$

K : **Static (Steady State, DC) Gain** $\quad K = \frac{b_0}{\omega_n^2} = \frac{b_0}{a_0}$

Poles of Stable 2nd Order Systems

- **Stable 2nd Order Systems without Zeros**

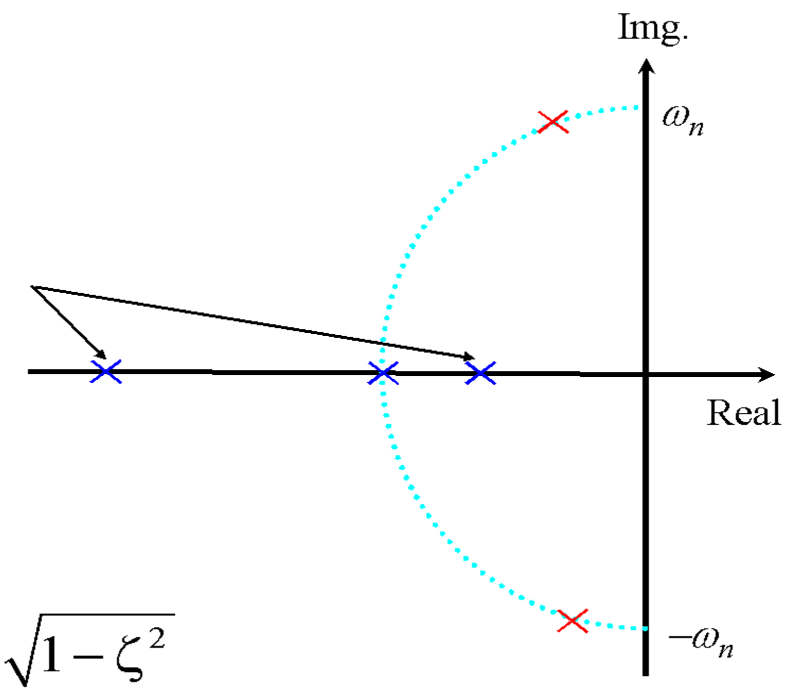
$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K\omega_n^2 u$$

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

- **Pole Locations**

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\Rightarrow \quad p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{(\zeta^2 - 1)}$$



– Over-damped ($\zeta > 1$)

Two distinct real poles

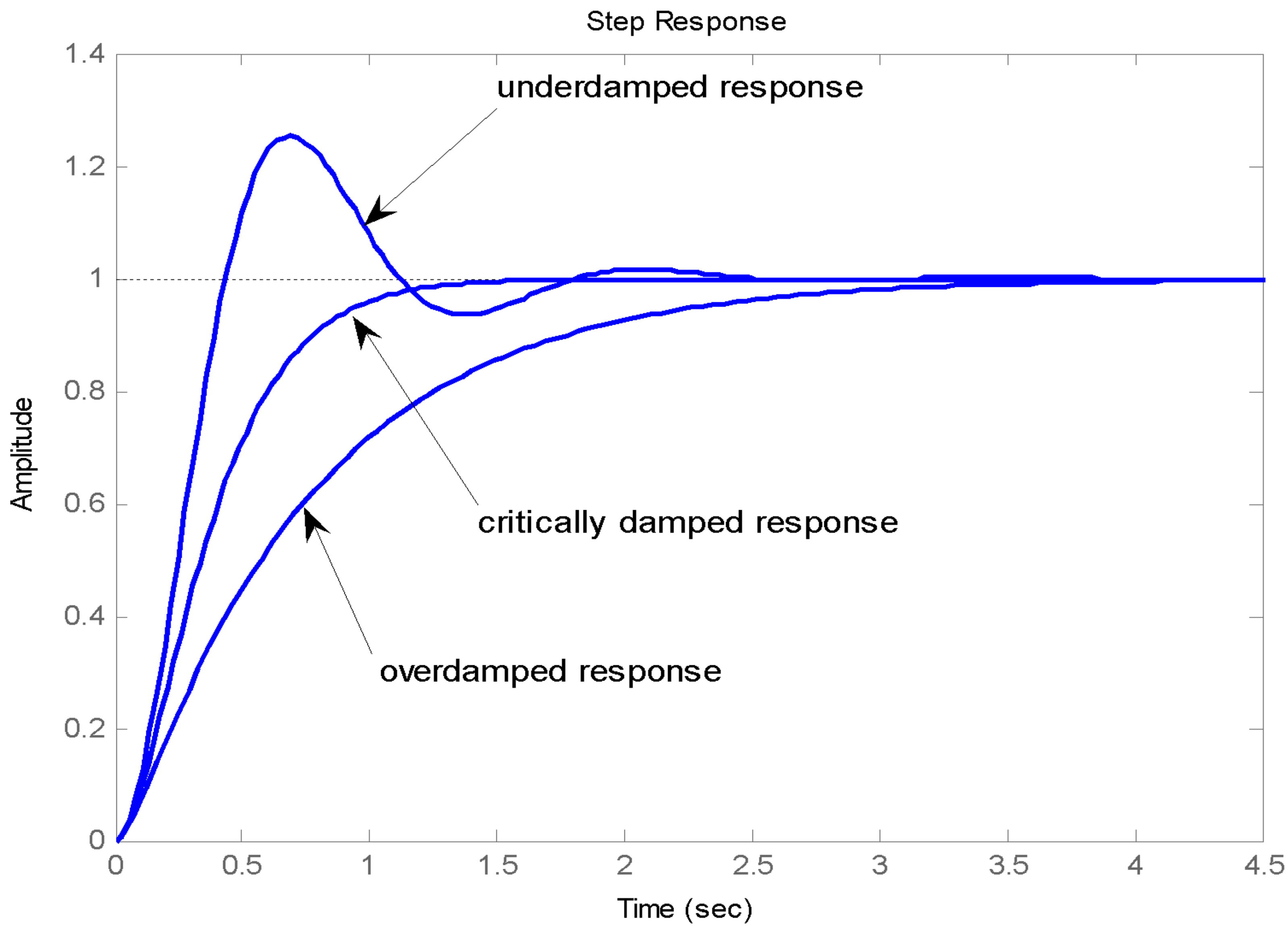
– Critically damped ($\zeta = 1$)

Two identical real poles at $p_{1,2} = -\zeta\omega_n$

– Under damped ($\zeta < 1$)

Two complex poles at $p_{1,2} = -\underbrace{\zeta\omega_n}_{\sigma} \pm j\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$

Time response of 2nd Order Systems



Under-damped 2nd Order System

- **Unit Step Response** (u=1 and zero ICs)

$$Y(s) = G(s)\frac{1}{s} = \frac{K\omega_n^{\ 2}}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}\right)} = \frac{K\omega_n^{\ 2}}{s(s-p_1)(s-p_2)}, \quad p_{1,2} = -\sigma \pm j\omega_d$$

$$= \frac{K}{s} + \frac{A_1}{s+\sigma - j\omega_d} + \frac{\bar{A}_1}{s+\sigma + j\omega_d}, \qquad A_1 = -\frac{K}{2}\left(1 - j\frac{\sigma}{\omega_d}\right)$$

$$\Rightarrow \quad y(t) = K + \underbrace{A_1 e^{(-\sigma + j\omega_d)t} + \bar{A}_1 e^{(-\sigma - j\omega_d)t}}_{2\,\mathrm{Re}\left\{A_1 e^{(-\sigma + j\omega_d)t}\right\}}$$

$$= K\left[1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\varsigma\omega_n}{\omega_d}\sin(\omega_d t)\right)\right]$$

$$= K - \frac{K}{\sqrt{1-\varsigma^2}} e^{-\sigma t}\sin(\omega_d t + \varphi), \qquad \varphi = \tan^{-1}\left(\frac{\sqrt{1-\varsigma^2}}{\varsigma}\right)$$

Under-damped 2nd Order System

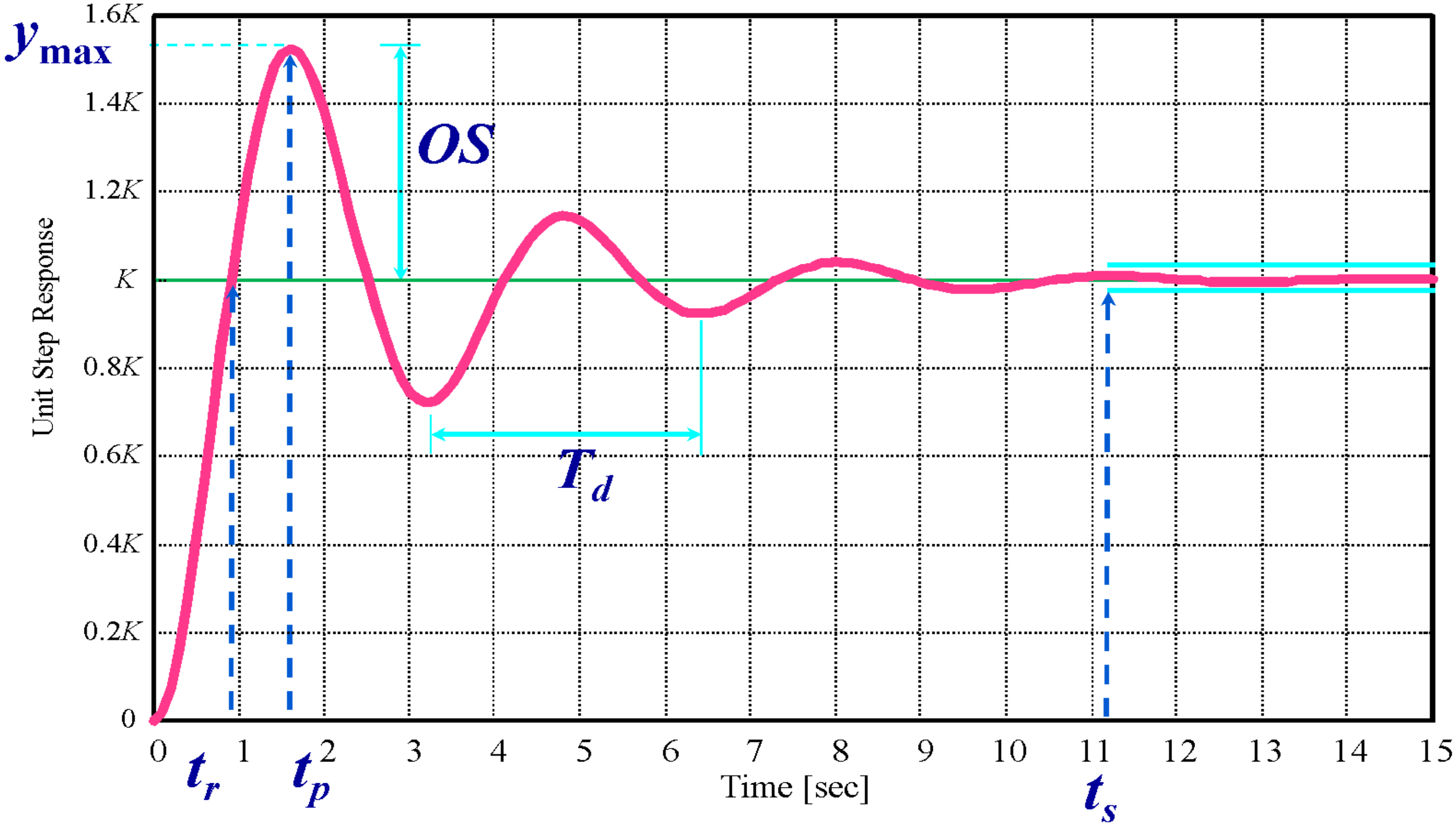
- **Unit Step Response** (u=1 and zero ICs)

$$Y(s) = G(s)\frac{1}{s} = K\left[\frac{\omega_n^{\ 2}}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}\right)}\right]$$

From Laplace Table:

$$y(t) = K\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\sigma t}\sin\left(\omega_d t + \varphi\right)\right], \qquad \varphi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Unit Step Response of 2nd Order Systems



Unit Step Response of 2nd Order System

- ## Peak Time (t_P)

Time when output $y(t)$ reaches its maximum value y_{MAX}.

$$\Rightarrow \quad y(t) = K\left[1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\varsigma\omega_n}{\omega_d}\sin(\omega_d t)\right)\right]$$

$$\frac{dy}{dt} = Ke^{-\varsigma\omega_n}\frac{\omega_n}{\sqrt{1-\varsigma^2}}\sin(\omega_d t)$$

Find t_P such that $\dot{y}(t_P) = 0$

$$\omega_d t_P = \pi$$

$$\Rightarrow$$

$$t_P = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$T_d = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}$$

- ## Percent Overshoot (%OS)

At peak time t_P the maximum output

$$y_{MAX} = y(t_P) = K\left(1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}}\right)$$

The overshoot (OS) is:

$$OS = y_{MAX} - y_{SS} = Ke^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}}$$

The percent overshoot is:

$$\%OS = \left(\frac{OS}{y_{SS} - y(0)}\right)100\%$$

$$= e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}} \times 100\%$$

Unit Step Response of 2nd Order System

- **Settling Time (t_s)**

  Time required for the response to be within a specific percent of the final (steady-state) value.

$$t_s = 3T = \frac{3}{\zeta\omega_n} \quad \text{5\% criterion is used}$$

$$t_s = 4T = \frac{4}{\zeta\omega_n} \quad \text{2\% criterion is used}$$

Q: Which parameters of a 2nd order system affect the peak time?

$$t_P = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Damping ration and natural frequency

Q: Which parameters of a 2nd order system affect the % OS?

Damping ratio $\quad \%OS = e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}} \times 100\%$

Q: Which parameters of a 2nd order system affect the settling time?

Damping ratio and natural frequency

Q: Can you obtain the formula for a 3% settling time?

Unit Step Response of 2nd Order System

Rise Time (t_r):

Time required for the response to rise to its final value. For underdamped second order systems the 0% to 100% rise time is used. For overdamped systems the 10% to 90% rise time is commonly used.

$$c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2}\, t_r + \phi) = 1$$

$$-\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2}\, t_r + \phi) = 0$$

$$\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \neq 0$$

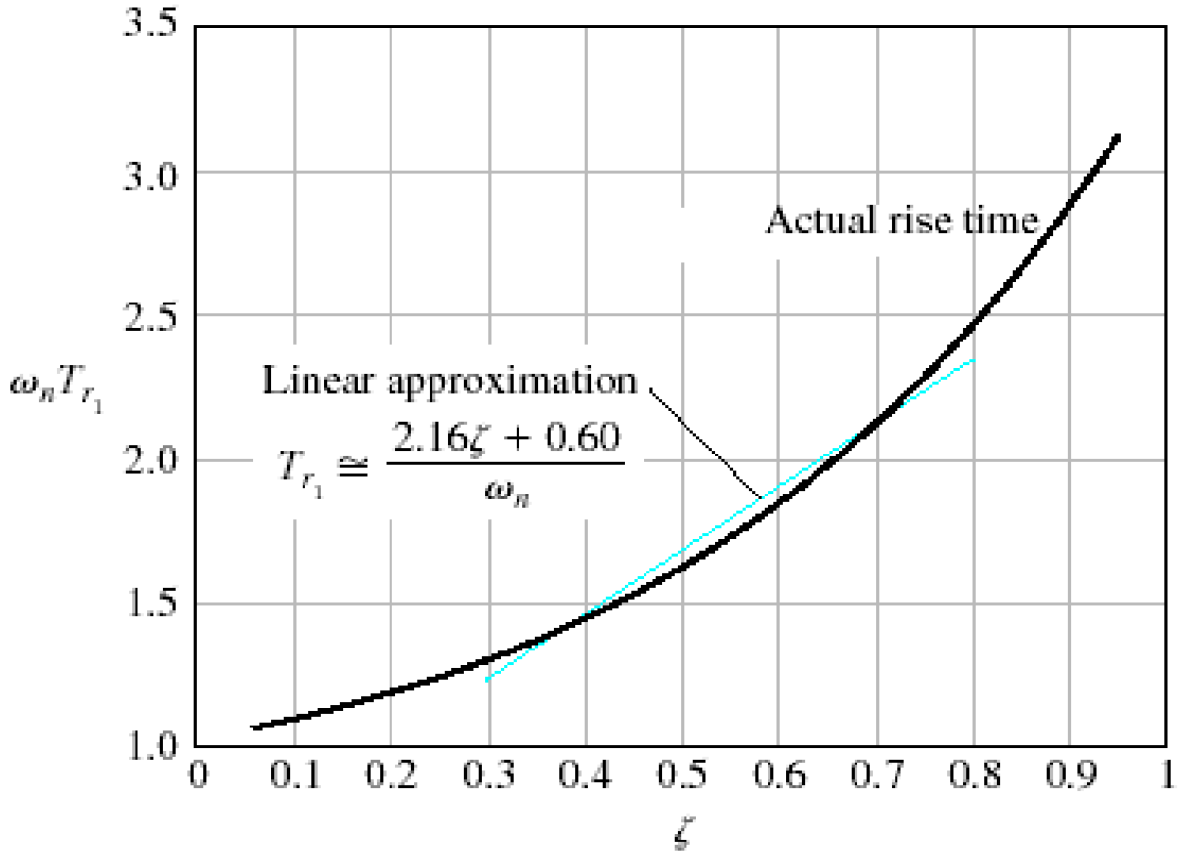
$$-\sin(\omega_n \sqrt{1-\zeta^2}\, t_r + \phi) = 0$$

$$\omega_n \sqrt{1-\zeta^2}\, t_r + \phi = \pi$$

$$t_r = \frac{\pi - \phi}{\omega_n \sqrt{1-\zeta^2}}$$

$$\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$$

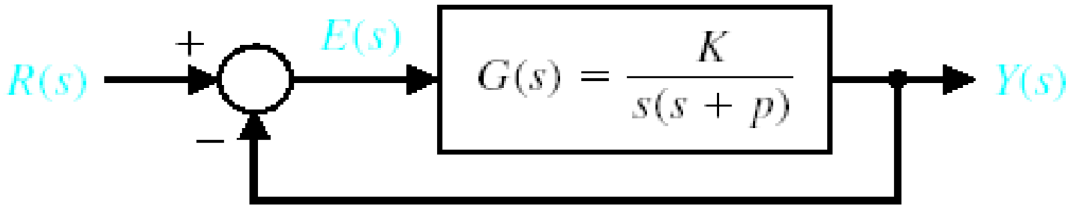
Performance of a Second-Order System



Normalized rise time T_{r_1} versus ζ for a second-order system.

Performance of a Second-Order System

$$Y(s) = \frac{K}{s^2 + p\cdot s + K}\cdot R(s)$$



$$Y(s) = \frac{\omega_n^2}{s^2 + 2\cdot\zeta\cdot\omega_n\cdot s + \omega_n^2}$$

with a unity step input

$$\beta = \sqrt{1 - \zeta^2}$$

$$\theta = \cos^{-1}(\zeta)$$

$$Y(s) = \frac{\omega_n^2}{\left(s^2 + 2\cdot\zeta\cdot\omega_n\cdot s + \omega_n^2\right)\cdot s}$$

$$y(t) = 1 - \frac{1}{\beta}\cdot e^{-\zeta\cdot\omega_n\cdot t}\cdot\sin\left(\omega_n\cdot\beta\cdot t + \theta\right)$$

Performance of a Second-Order System

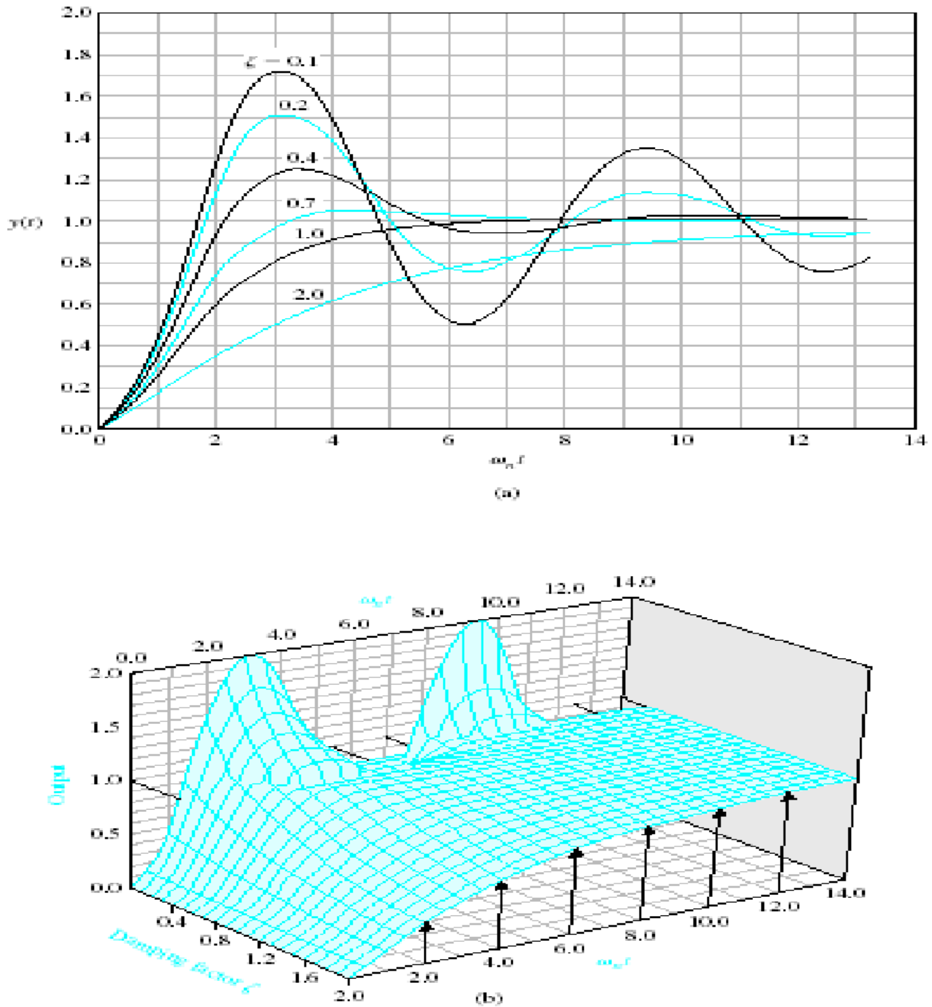
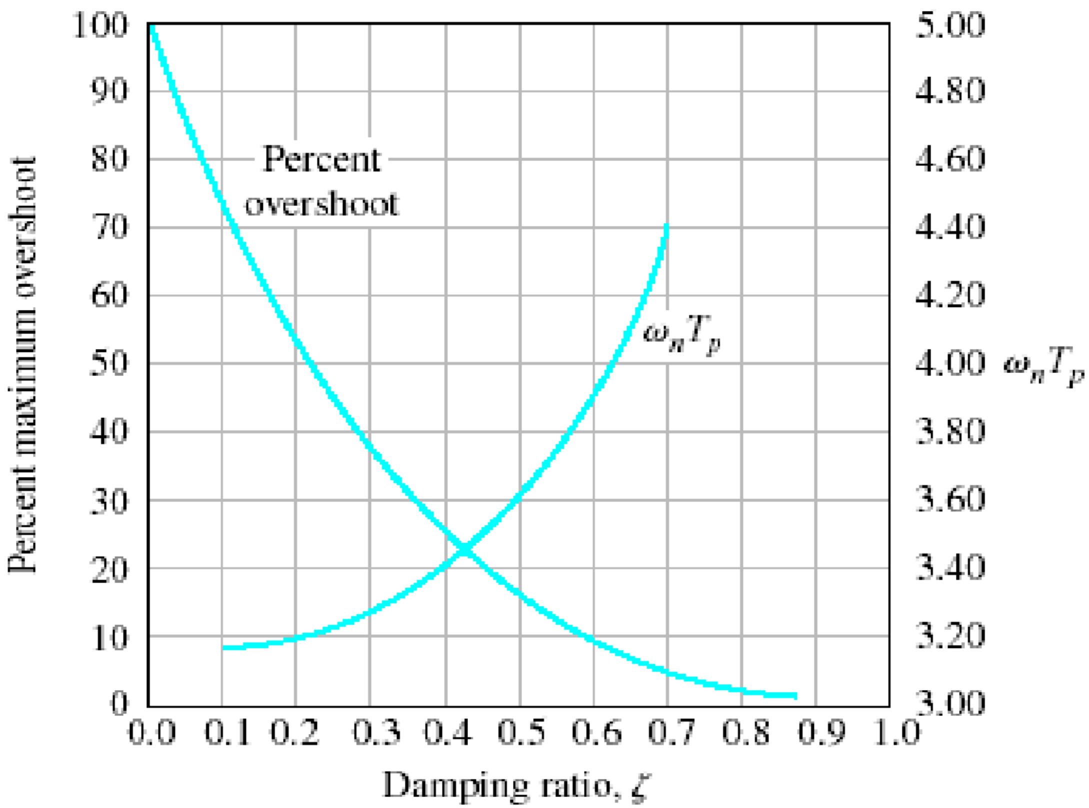


FIGURE 5.5

(a) Transient response of a second-order system (Eq. 5.9) for a step input. (b) The transient response of a second-order system (Eq. 5.9) for a step input as a function of ζ and $\omega_n t$. (Courtesy of Professor R. Jacquot, University of Wyoming.)

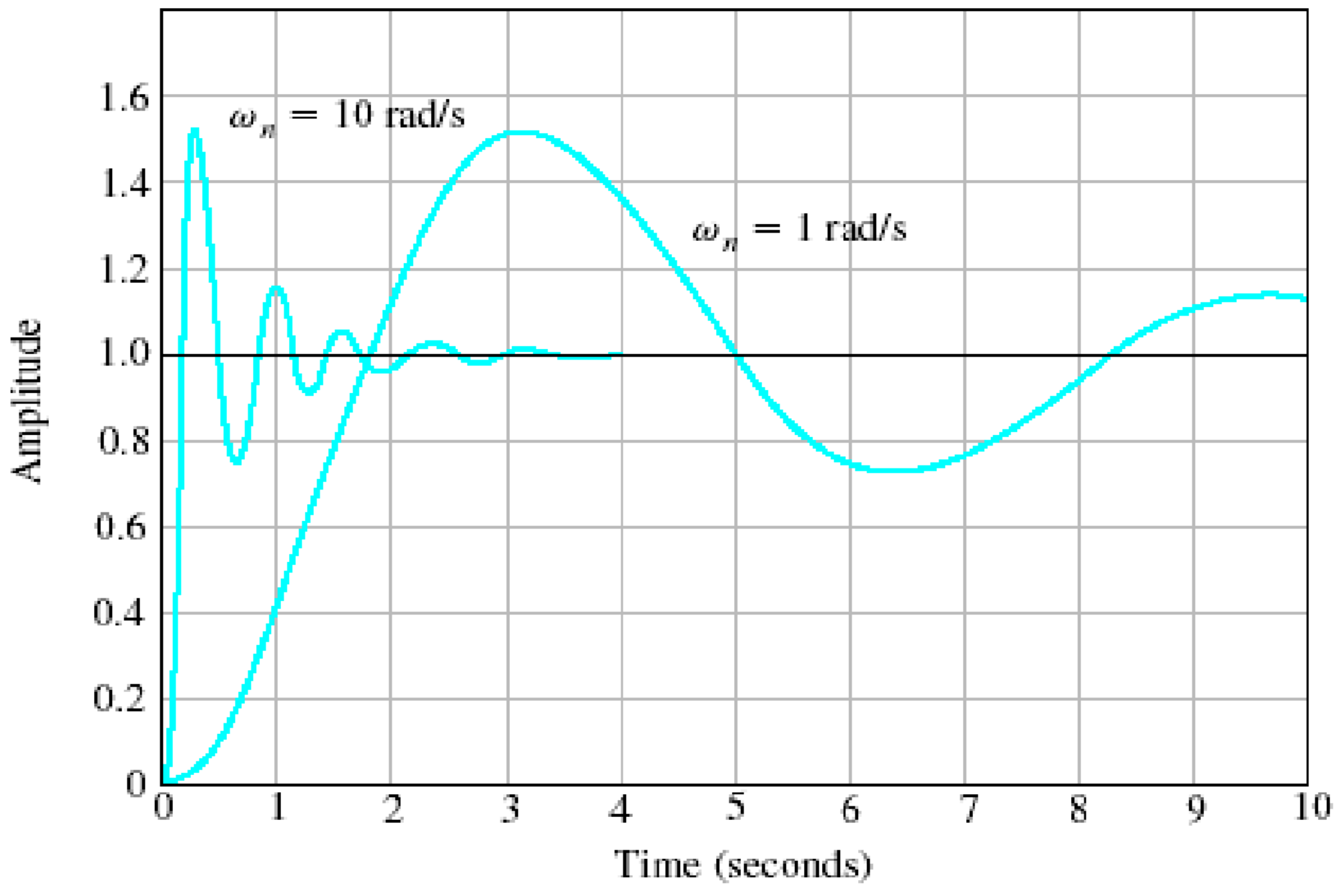
Performance of a Second-Order System



Percent overshoot and normalized peak time versus damping ratio ζ for a second-order system (Eq. 5.8).

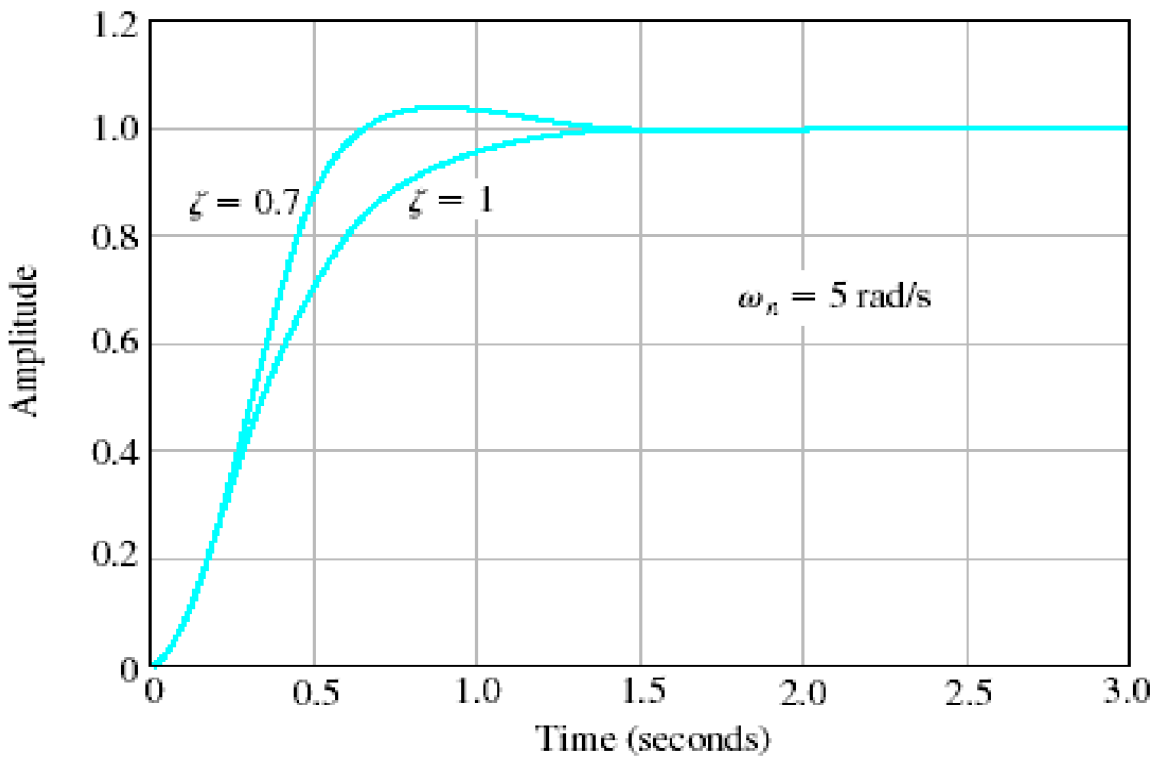
Naturally these two performance measures are in opposition and a compromise must be made.

Performance of a Second-Order System



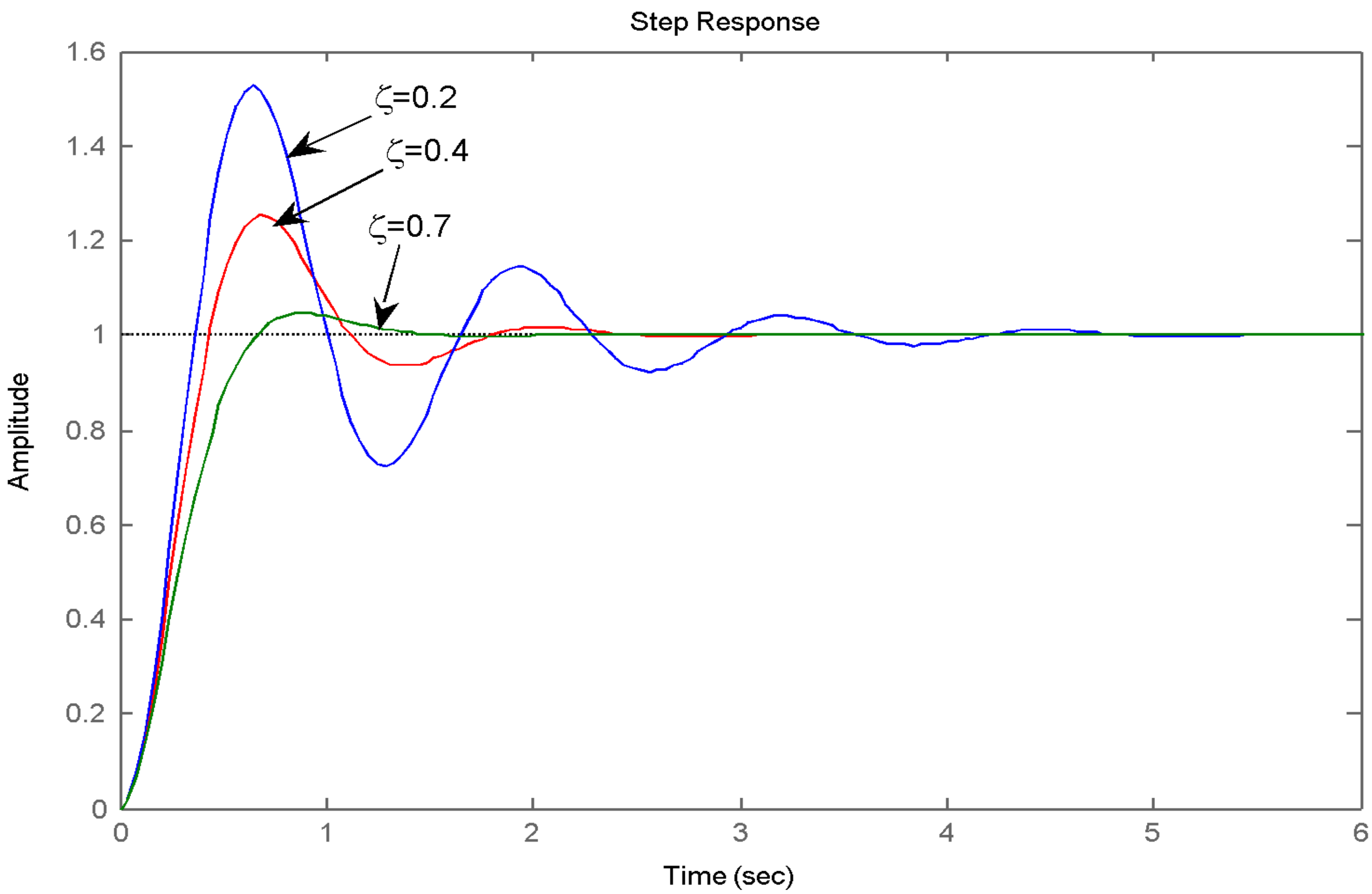
The step response for $\zeta = 0.2$ for $\omega_n = 1$ and $\omega_n = 10$.

Performance of a Second-Order System



The step response for $\omega_n = 5$ with $\zeta = 0.7$ and $\zeta = 1$.

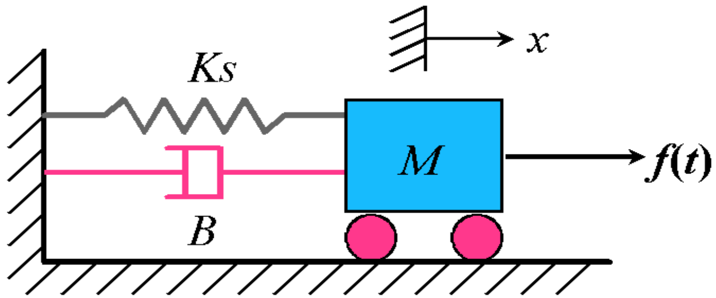
Performance of a Second-Order System



Response of a second-order system for an impulse function input.

In Class Exercise 1

- **Mass-Spring-Damper System**



I/O Model:

Q: *What is the static gain of the system ?*

Q: *How would the physical parameters (M, B, K) affect the step response of the system ?*

(This is equivalent to asking you for the relationship between the physical parameters and the damping ratio, natural frequency and the static gain.)

$$\ddot{y} + a_1\dot{y} + a_0 y = b_0\, u \quad \Rightarrow \quad \ddot{y} + \underbrace{2\zeta\omega_n}_{a_1}\dot{y} + \underbrace{\omega_n^{\,2}}_{a_0}\, y = \underbrace{K\omega_n^{\,2}}_{b_0} u$$

ω_n : **Natural Frequency [rad/s]**

ζ : **Damping Ratio** $\quad \varsigma = \dfrac{a_1}{2\omega_n} = \dfrac{a_1}{2\sqrt{a_0}}$

K : **Static (Steady State, DC) Gain** $\quad K = \dfrac{b_0}{\omega_n^{\,2}} = \dfrac{b_0}{a_0}$

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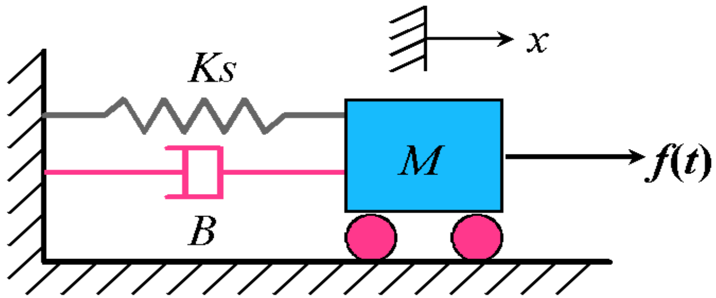
In Class Exercise 1

- **Mass-Spring-Damper System**



I/O Model:

$$M\ddot{x} + B\dot{x} + K_s x = f(t)$$

$$G(s) = \frac{1}{Ms^2 + Bs + K_s} = \frac{1/M}{s^2 + \frac{B}{M}s + \frac{K_s}{M}} = \frac{K\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

$$\omega_n = \sqrt{\frac{K_s}{M}}$$

$$\varsigma = \frac{B}{2\sqrt{MK_s}}$$

$$K = \frac{1}{K_s}$$

Q: *What is the static gain of the system ?*

Q: *How would the physical parameters (M, B, K) affect the step response of the system ?*

(This is equivalent to asking you for the relationship between the physical parameters and the damping ratio, natural frequency and the static gain.)

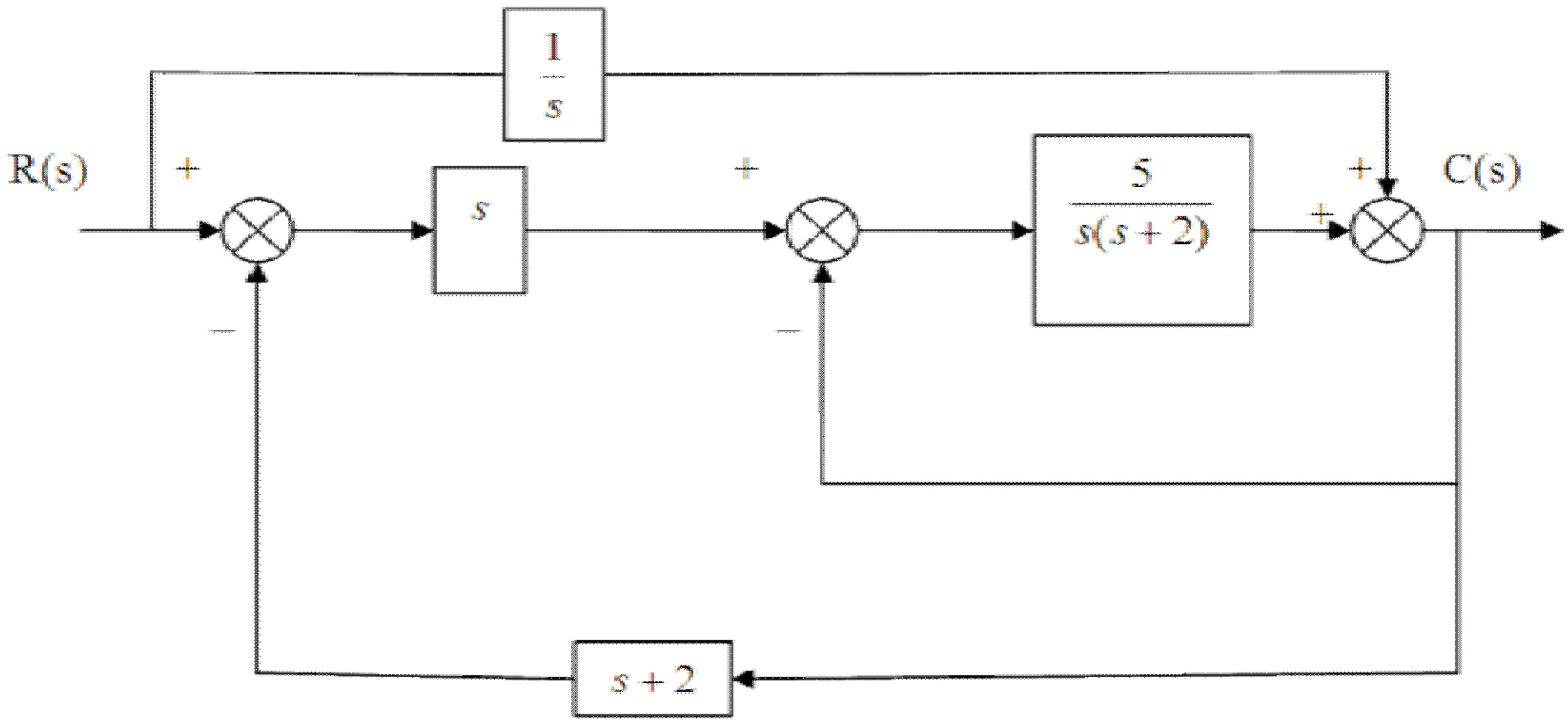
In Class Exercise 2

Find for the given control system block diagram:

a) Show that Transfer function is: $\frac{C(s)}{R(s)} = \frac{6s+2}{6s^2+12s+5}$

b) Peak time and OS

c) Static gain, K



In Class Exercise 2

$$\frac{C(s)}{R(s)} = \frac{s\frac{5}{s(s+2)} + \frac{1}{s}}{1 + s\frac{5}{s(s+2)}(s+2) + \frac{5}{s(s+2)}} = \frac{5s + (s+2)}{s(s+2) + 5s(s+2) + 5} = \frac{6s+2}{6s^2 + 12s + 5}$$