# MENG366

# Routh's Stability Criterion

## Part I

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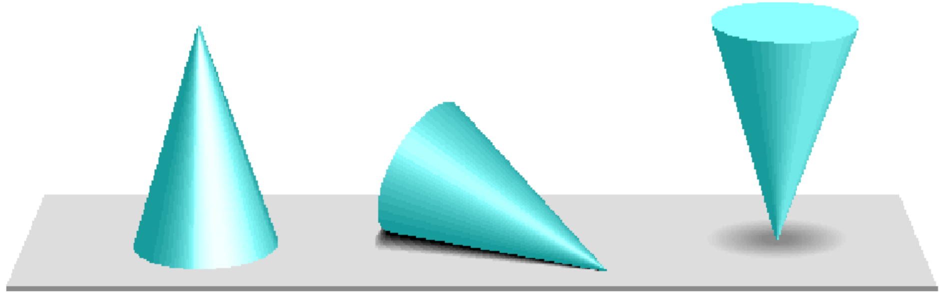
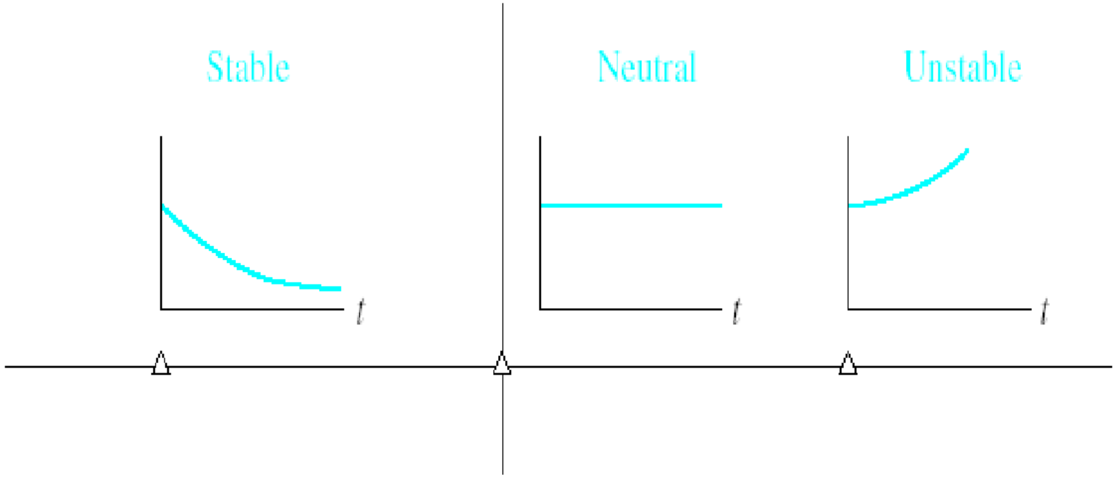
# The Concept of Stability

The concept of stability can be illustrated by a cone placed on a plane horizontal surface.



(a) Stable (b) Neutral (c) Unstable



A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

A system is considered marginally stable if only certain bounded inputs will result in a bounded output.

# Stability

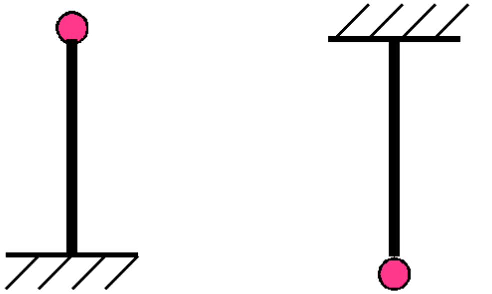
where the derivatives of all states are zeros

- **Stability Concept**

  Describes the ability of a system to stay at its <u>equilibrium position</u> in the absence of any inputs.

  - **A linear time invariant (LTI) system is stable if and only if (iff) its free response converges to zero for all ICs.**

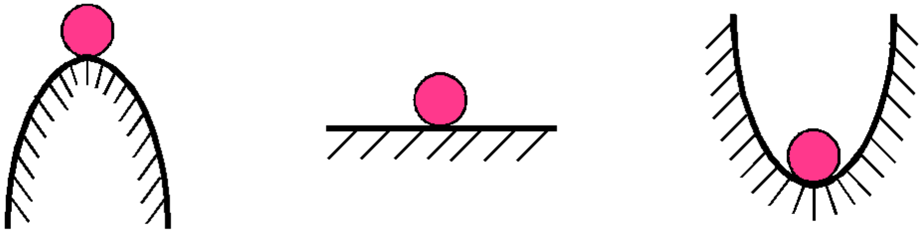
Ex: Pendulum



inverted pendulum

simple pendulum

Ball on curved surface



hill

plateau

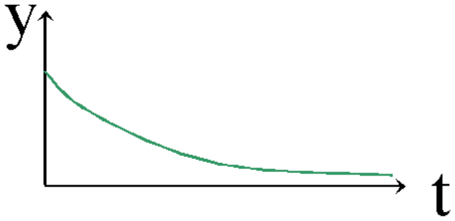
valley

# Examples (stable and unstable 1st order systems)

**Q: *free response of a 1st order system.***

$$5\dot{y} + y = u(t) \qquad y(0) = y_0$$

$$\boxed{y(t) = y_0 e^{-\frac{1}{5}t}}$$



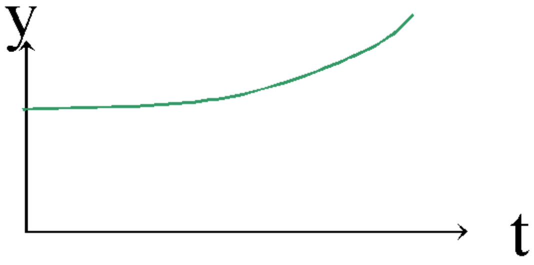
TF: $G = \dfrac{1}{5s+1}$

Pole: $p = -0.2$

**Q: free response of a 1st order system.**

$$-5\dot{y} + y = u(t) \qquad y(0) = y_0$$

$$\boxed{y(t) = y_0 e^{\frac{1}{5}t}}$$



TF: $G = \dfrac{1}{-5s+1}$

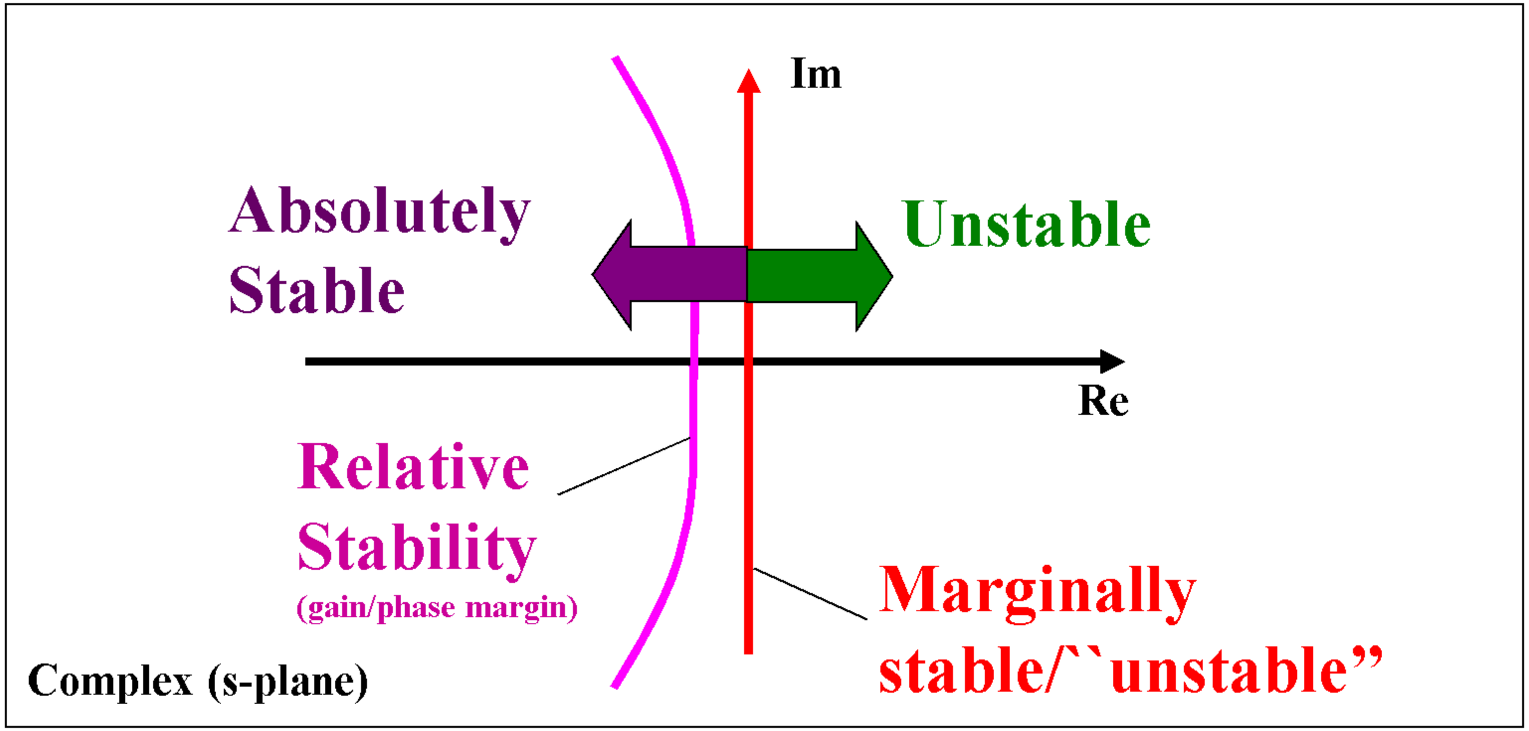
Pole: $p = 0.2$

# Stability of LTI Systems

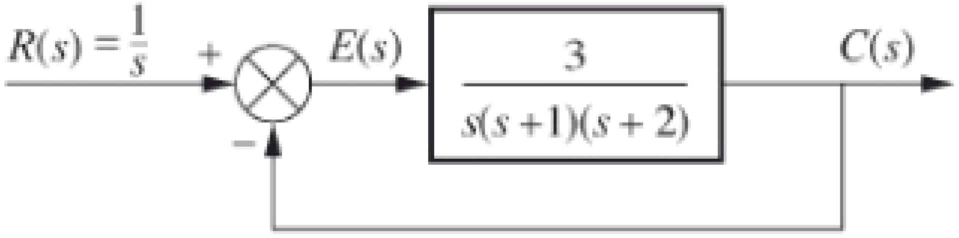
$$y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1\dot{y}+a_0y=b_mu^{(m)}+b_{m-1}u^{(m-1)}+\cdots+b_1\dot{u}+b_0u$$

Stable $\Longleftrightarrow$ All poles lie in the left-half complex plane (LHP)

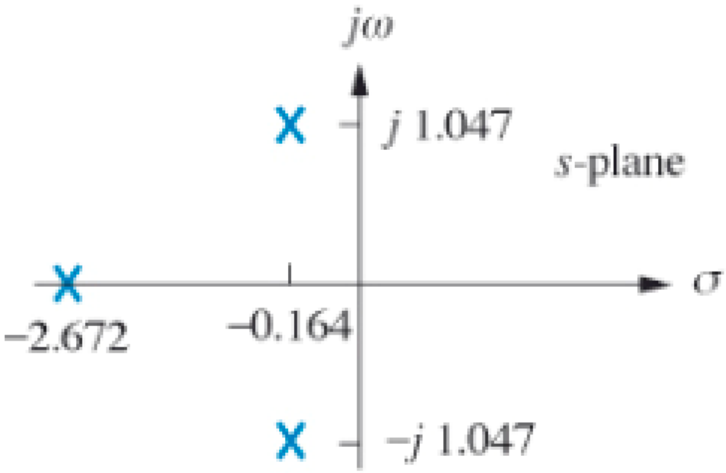
$\Longleftrightarrow$ All roots of $D(s)=\underbrace{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}_{\text{Characteristic Polynomial}}=0$ lie in the LHP



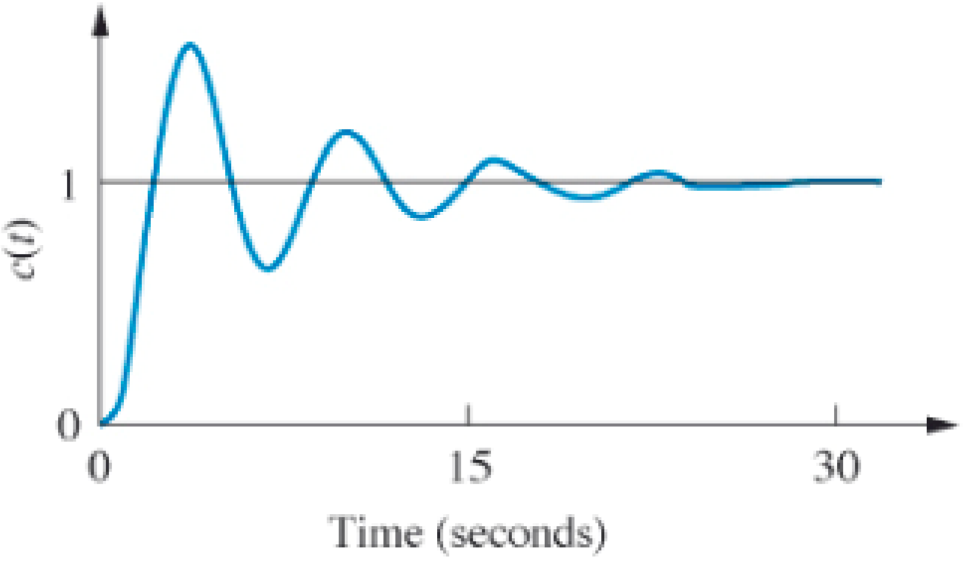
# Stable System



Stable system



Stable system's
closed-loop poles
(not to scale)



# Stable System

**Closed –Loop TF**

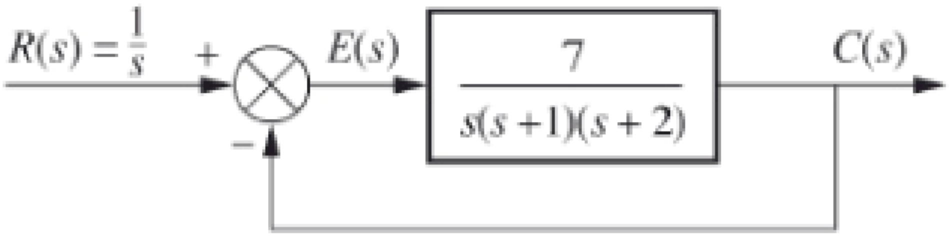
$$G = \frac{3}{s^3 + 3s^2 + 2s + 3}$$

**Poles**

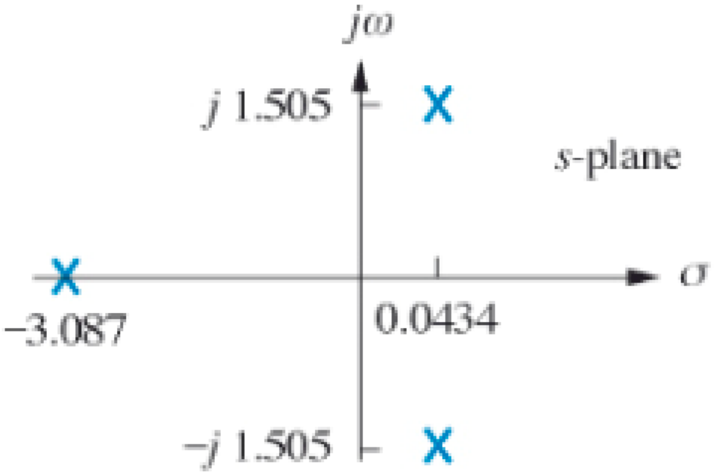
```
>> p=[1 3 2 3];
>> roots(p)

ans =


  -2.6717
  -0.1642 +
1.0469i
  -0.1642 -
1.0469i
```

➡ **All in LHS of s- plane**

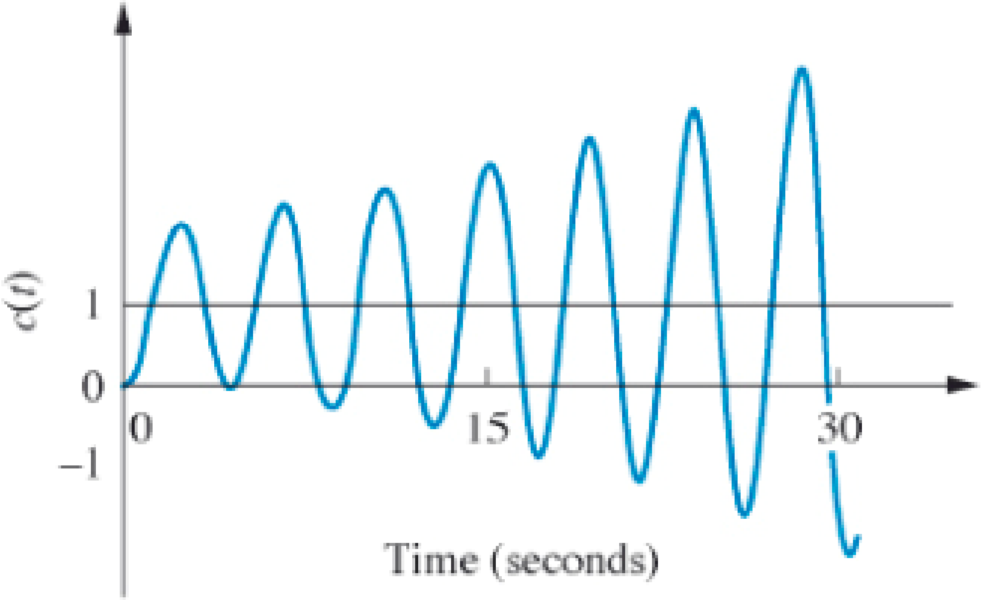
# Unstable System



Unstable system



Unstable system's closed-loop poles (not to scale)



# Unstable System

**Closed –Loop TF**

$$G = \frac{7}{s^3 + 3s^2 + 2s + 7}$$

**Poles**

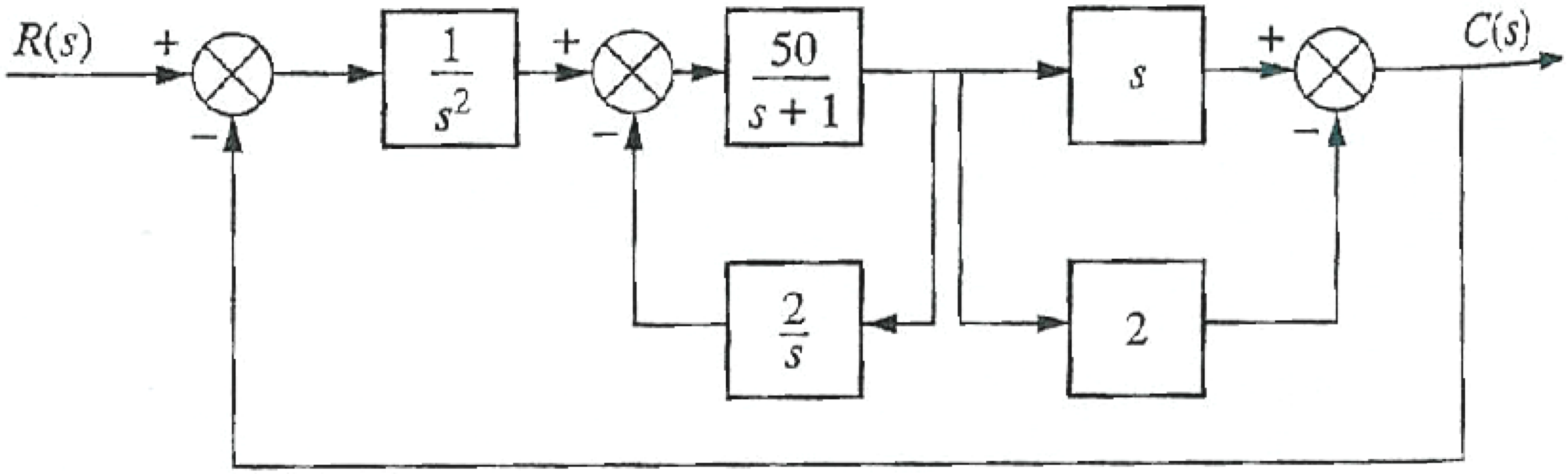
```
>> p=[1 3 2 7];
>> roots(p)

ans =


   -3.0867
   0.0434 +
1.5053i
   0.0434 -
1.5053i
```

→ **One in LHS of s- plane**

**Two in RHS**

# Unstable System

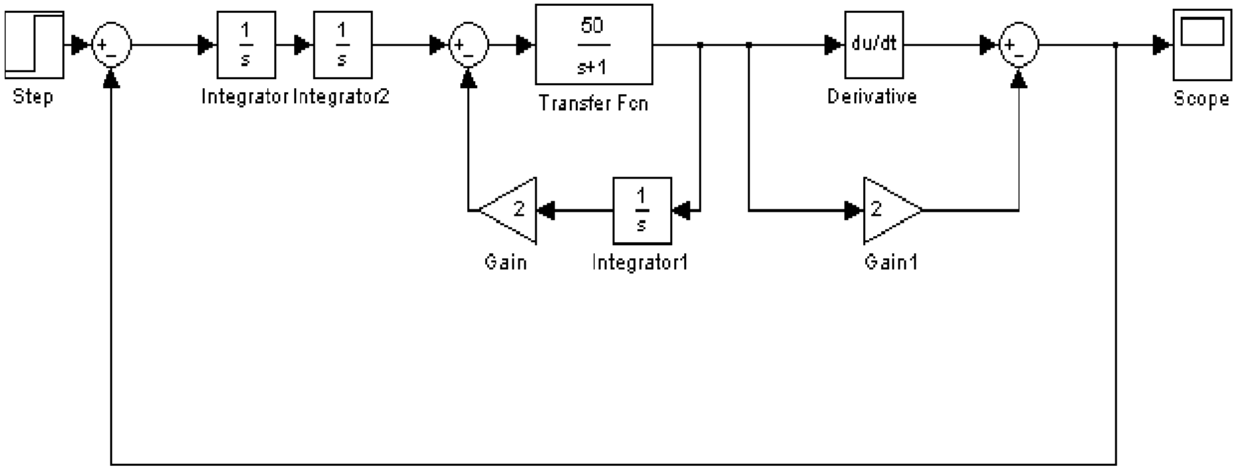
## Block Diagram

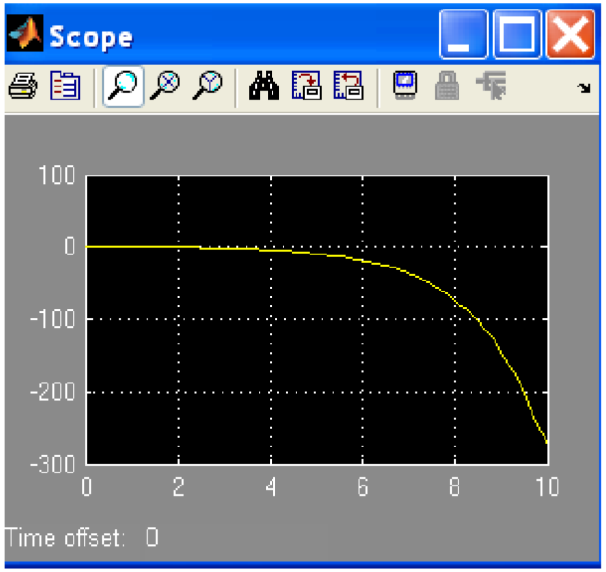


$$T(s) = \frac{50(s-2)}{s^3+s^2+150s-100}$$

# Unstable System

## Simulink model

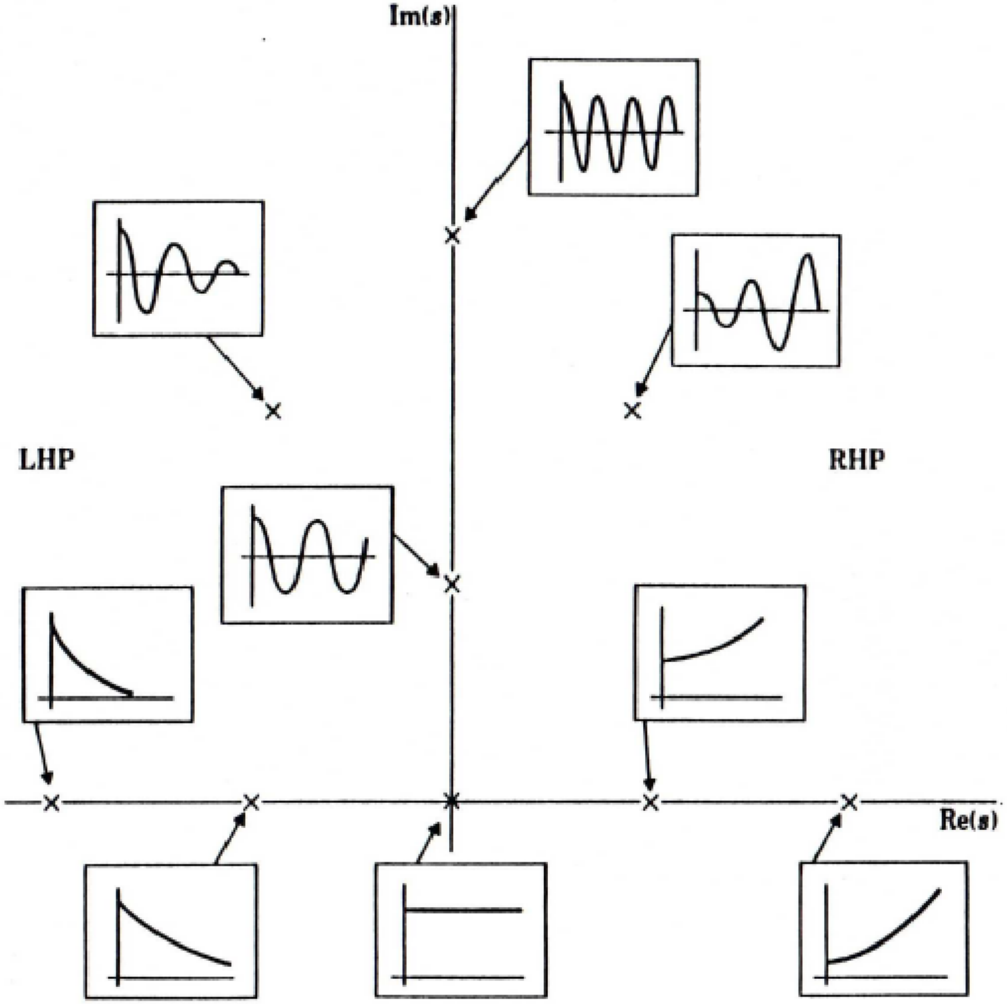




**Unstable as** $$T(s) = \frac{50(s-2)}{s^3+s^2+150s-100}$$

**has poles at** $-0.8309 \pm 12.2642i, \boxed{0.6618}$

# Stable & Unstable System



# Stability Definitions

- **Bounded Input Bounded Output Stability:**

  **BIBO** system is stable if, for every bounded input, the output remains bounded with increasing time (**all system poles must lie in the left half of the s-plane**).

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- **Marginal Stability:**

  Occurs if some of the poles lie on the **imaginary axis**, while all others are in the LHS of the s-plane.

# Internal (Asymptotic) Stable

***Definition*** :

*A system is internal (asymptotic) stable, if the zero-input response decays to zero, as time approaches infinity, for all possible initial conditions.*

*Asymptotic stable =>*All the characteristic polynomial roots are located in the LHP (left-half-plan)

# External (BIBO) Stable

***Definition*** :

*A system is external (bounded-input, bounded-output, BIBO) stable, if the zero-state response is bounded, as time approaches infinity, for all bounded inputs.*

*bounded-input, bounded-output stable* =>All the poles of transfer function are located in the LHP (left-half-plan)

*Asymptotic stable => BIBO stable*
*BIBO stable ≠> Asymptotic stable*

# Poles and zeros

- Poles: values of s at which TF → infinity
  - Most time, poles = roots of denominator
  - When there are common factors in numerator and denominator, cancel them first
- Zeros: values of s at which TF = 0
  - Finite zeros: roots of numerator
  - Number of zeros at infinity: n-m, n= den deg and m = num deg
- Totally n poles and n zeros
- n is called the order of the system
- n – m the relative order

# Example

$$G(s) = \frac{10(s+1)}{s^2(s+4)(s+6)}$$

- Order: n = 4 = den deg
- m= num deg = 1
- Relative order = n-m = 3
- 4 poles at: 0, 0, -4, -6
- One finite zero at -1
- 3 zeros at infinity

# BIBO Stability

- System is BIBO stable if any bounded input generates bounded output
- Simple criteria:
  - After common factor cancellation
  - All poles have strictly negative real parts

# BIBO Stability

**POP. Quiz:**

$$G_1(s) = \frac{10(s+1)}{s^2(s+4)(s+6)}$$

$$G_2(s) = \frac{10s+3}{(s+2)(s+5)}$$

$$G_3(s) = \frac{s-1}{s^2+4s+6}$$

$$G_4(s) = \frac{e^{-2s}}{10s(s+1)(s+2)}$$

# BIBO Stability

**POP. Quiz:**

$$G_1(s)=\frac{10(s+1)}{s^2(s+4)(s+6)}$$

$$G_2(s)=\frac{10s+3}{(s+2)(s+5)}$$

$$G_3(s)=\frac{s-1}{s^2+4s+6}$$

$$G_4(s)=\frac{e^{-2s}}{10s(s+1)(s+2)}$$

G1 and G4 are not BIBO stable
G2 and G3 are BIBO stable

# Stability Analysis

## Methods for Testing Stability of a LTI system

1. Examine the poles of the system.

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2. Methods the do not require the actual solution of the characteristic equation.

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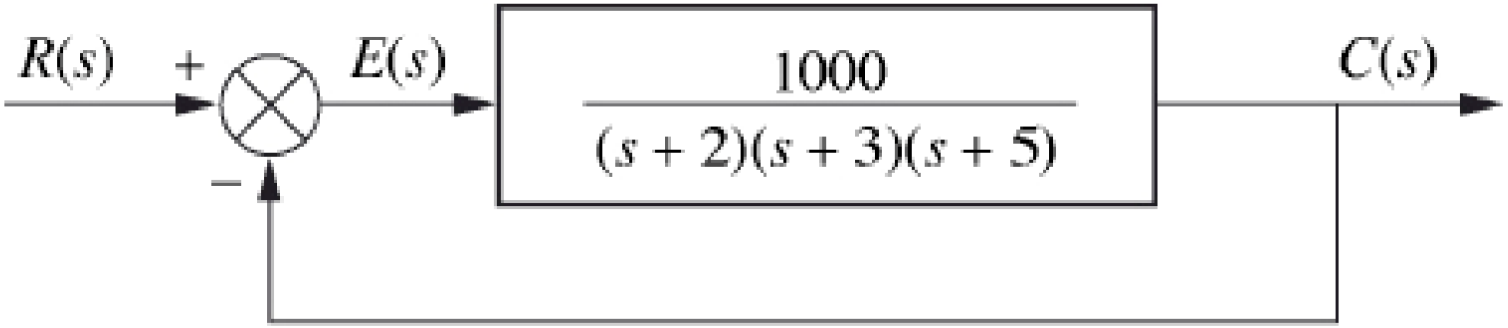
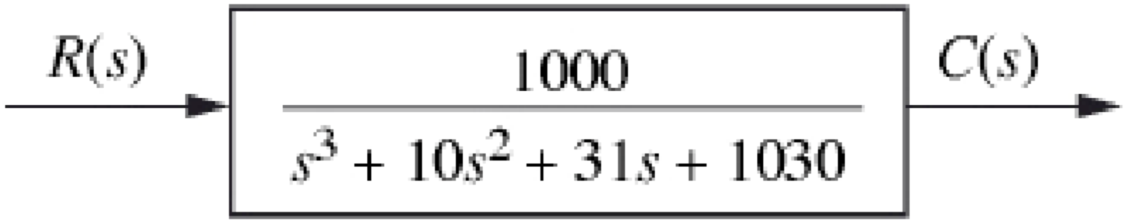
3. Methods that are based only on the loop transfer function characteristics.

# Poles of Complex Systems



**Open-loop poles**: s= -2, -3, -5



**Closed-loop poles**: -13.4136, 1.7068 + 8.5950*j*, 1.7068 - 8.5950*j*

# Routh-Hurwitz Criterion

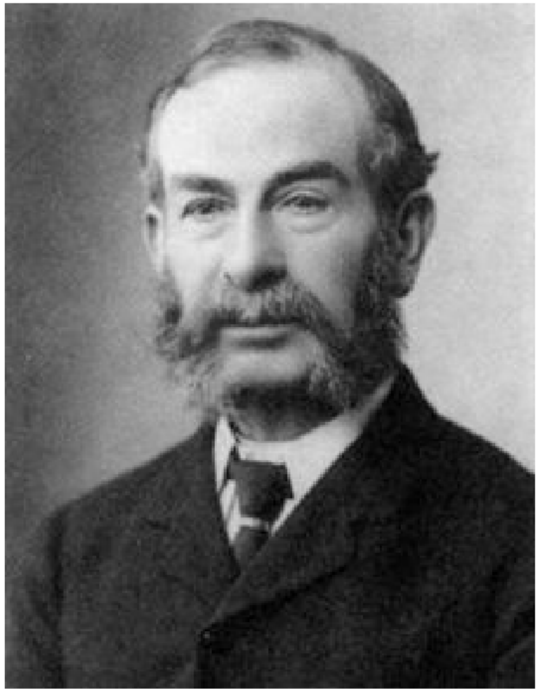
Is a method that checks system stability without actual solution of the characteristic equation.

Is a method for establishing bounds on system parameters to ensure stability.

**Nuoki Mutsumoto,** ‘Simple Proof of The Routh Stability Criterion Based on Order Reduction of Polynomials and Principle of Argument”, *The 2001 IEEE International Symposium on Circuits and Systems*, 6-9 May, 2001, Sydney Australia, Vol. I, pp. 699-702

# E. Routh & A. Hurwitz

**Edward John Routh**
**(1831-1907)**



University of Cambridge

**Adolf Hurwitz**
**(1859 -1919)**

Eidgenössische Polytechnikum
Zürich

# Routh-Hurwitz Stability Criterion

## A quick method for checking BIBO stability

- Assume the characteristic polynomial is

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

where $a_0 \neq 0$.

- A **necessary** (but not sufficient) condition for all roots to have non-positive real parts is that **all coefficients have the same sign.**
- For the **necessary and sufficient** conditions, the sign of the first column of the **Routh array** should not change.

# The Routh Array

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

$$\begin{array}{ccccc}
a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
b_1 & b_2 & b_3 & b_4 & \cdots \\
c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & \\
k_1 & k_2 & & & \\
l_1 & & & & \\
m_1 & & & &
\end{array}$$

where

$$b_1 = \frac{a_{n-1} a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1} a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

etc.

# The Routh Array

- In a similar manner, elements in the 4th row, $c_1$, $c_2$ , … are calculated based as:

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$$

$$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

- The elements in all subsequent rows are calculated in the same manner.

# Routh-Hurwitz Stability Criterion

## Necessary and sufficient conditions:

- If all elements in the **first column** of the Routh array have the **same sign**, then all roots of the characteristic equation have negative real parts.
- If there are sign changes in these elements, then the number of roots with non-negative real parts is equal to the **number of sign changes**.
- Elements in the first column which are **zero** define a special case.

# Proof of Routh-Hurwitz Criterion

## Second Order Systems

$$Q(s) = s^2 + as + b = 0$$

**with *a, b* real & positive**

## Routh Table

$$\begin{array}{c|cc} s^2 & 1 & b \\ s^1 & a & 0 \\ s^0 & b_1 & \end{array} \longrightarrow b_1 = \frac{a \times b - 1 \times 0}{a} = b$$

## Necessary Condition

System is stable if ***a & b*** are positive, i.e. no sign changes

# Proof of Routh-Hurwitz Criterion

**Note that the roots of the CE are:**

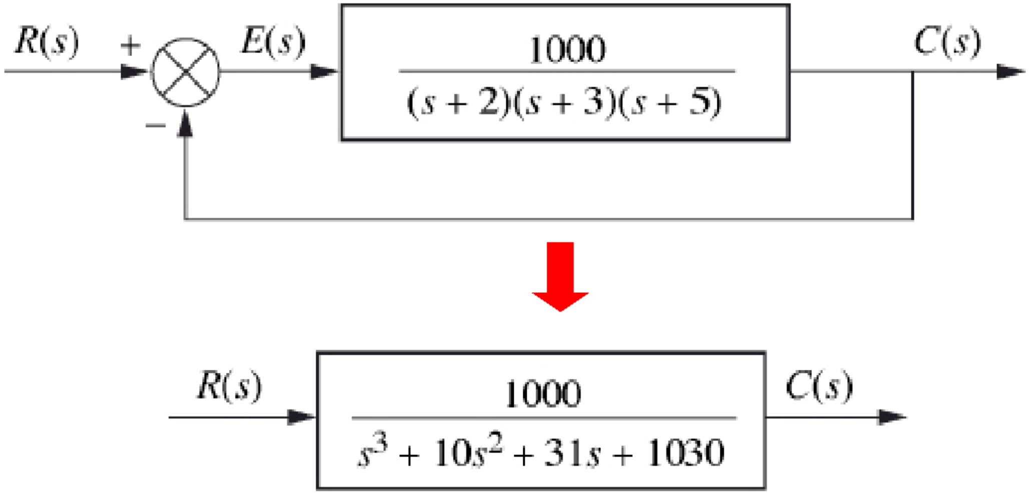
$$s_{1,2} = \frac{-a \pm \sqrt{-a^2 + 4b}\, j}{2}$$

$$\Rightarrow \qquad s_1 + s_2 = -a, \;\; s_1 s_2 = b$$

## Sufficient Condition

**Or** ***a & b*** are positive only when the roots have negative real parts

# Example 6.1



| | | | |
|---|---|---|---|
| $s^3$ | 1 | 31 | 0 |
| $s^2$ | ~~10~~ 1 | ~~1030~~ 103 | 0 |
| $s^1$ | $\frac{-\begin{vmatrix}1 & 31\\ 1 & 103\end{vmatrix}}{1} = -72$ | $\frac{-\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}}{1} = 0$ | $\frac{-\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}}{1} = 0$ |
| $s^0$ | $\frac{-\begin{vmatrix}1 & 103\\ -72 & 0\end{vmatrix}}{-72} = 103$ | $\frac{-\begin{vmatrix}1 & 0\\ -72 & 0\end{vmatrix}}{-72} = 0$ | $\frac{-\begin{vmatrix}1 & 0\\ -72 & 0\end{vmatrix}}{-72} = 0$ |

# Example 6.1

| | | | | |
|---|---|---|---|---|
| $s^3$ | 1 | 31 | 0 | 0 |
| $s^2$ | 10 | 1030 | 0 | |
| $s^1$ | -72 | 0 | | |
| $s^0$ | 1030 | | | |

**Two sign changes → two roots with positive real parts**

```
>> p=[1 10  31
1030];
>> roots(p)

ans =
 -13.4136
   1.7068 + 8.5950i
   1.7068 - 8.5950i
```

# Example 6.1

```
>> Routh_H([1  10  31 1030])
--------------------------------------------------------------------------------
                              RESULTS
--------------------------------------------------------------------------------
   Not satisfied with the sufficient condition.
   The system is unstable and has 2 pole(s) in the RSP.
--------------------------------------------------------------------------------

 ROUTH-HURWITZ ARRAY

   s^3  |      1       |      31     |
   s^2  |      10      |    1030     |
   s^1  |     -72      |
   s^0  |     1030     |
```

http://www.mathworks.fr/matlabcentral/fileexchange/19872

# Routh-Hurwitz Stability Criterion

**Example 1** $Q(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$

| | | | | |
|---|---|---|---|---|
| $s^4$ | 2 | 3 | 10 | 0 |
| $s^3$ | 1 | 5 | 0 | 0 |
| $s^2$ | $b_1$ | $b_2$ | 0 | |
| $s^1$ | $c_1$ | 0 | | |
| $s^0$ | $d_1$ | | | |

$$b_1 = \frac{3-10}{1} = -7 \qquad b_2 = \frac{10-0}{1} = 10$$

$$c_1 = \frac{-35-10}{-7} = 6.43$$

$$d_1 = \frac{10(6.43)-0}{6.43} = 10$$

The characteristic equation has **two** roots with **positive real parts** since the elements of the first column have two sign changes. (**2,1,-7,6.43,10**)

# Routh-Hurwitz Stability Criterion

**Roots**

$$Q(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$$

```
>> p=[2 3 5 10];
>> roots(p)

ans =

  -1.7279
   0.1139 + 1.6973i
   0.1139 - 1.6973i
```

The characteristic equation has **two** roots with **positive real parts** since the elements of the first column have two sign changes. (**2,1,-7,6.43,10**)

# Routh-Hurwitz Stability Criterion

```
>> Routh_H([2  1  3  5 10])
--------------------------------------------------------------------------------
                              RESULTS
--------------------------------------------------------------------------------
   Not satisfied with the sufficient condition.
   The system is unstable and has 2 pole(s) in the RSP.
--------------------------------------------------------------------------------


 ROUTH-HURWITZ ARRAY

   s^4 |     2        |    3     |    10    |
   s^3 |     1        |    5     |
   s^2 |    -7        |    10    |
   s^1 |   6.4286     |
   s^0 |    10        |
```

# Routh-Hurwitz Stability Criterion

## Special Case 1

- A **zero** in the first column:
- Remedy: substitute **ε** for the zero element, finish the Routh array, and then let **ε → 0** .

$$Q(s) = s^3 - 3s + 2$$

|  |  |  |  |  |
|---|---|---|---|---|
| $s^3$ | 1 | -3 | 0 | 0 |
| $s^2$ | 0(ε) | 2 | 0 |  |
| $s^1$ | $b_1$ | 0 |  |  |
| $s^0$ | $c_1$ |  |  |  |

$$b_1 = \frac{-3\varepsilon - 2}{\varepsilon} \rightarrow \frac{-2}{\varepsilon} \text{(negative)}$$

$$c_1 = \frac{b_1 \times 2}{b_1} = 2$$

There are **two** roots with positive real parts (**1, ε, -2/ε , 2**)

# Routh-Hurwitz Stability Criterion

## Special Case 1

$$Q(s) = s^3 - 3s + 2$$

```
>> p=[1 0 -3 2];
>> roots(p)

ans =

  -2.0000
   1.0000
   1.0000
```

There are **two** roots with positive real parts (**1, ε, -2/ε , 2**)

# Routh-Hurwitz Stability Criterion

```
>> Routh_H([1 0 -3 2])
--------------------------------------------------------------------------
                         RESULTS
--------------------------------------------------------------------------
   Not satisfied with the necessary condition.
   Not satisfied with the sufficient condition.
   The system is unstable and has 2 pole(s) in the RSP.
--------------------------------------------------------------------------

 ROUTH-HURWITZ ARRAY

   s^3 |      1       |     -3      |
   s^2 |      0       |      2      |
   s^1 |    -Inf      |
   s^0 |      2       |
```

# Example on Routh Stability

**Determine the stability of the closed-loop transfer function**

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

# Example on Routh Stability

## Routh Table

| | | | |
|---|---|---|---|
| $s^5$ | $1$ | $3$ | $5$ |
| $s^4$ | $2$ | $6$ | $3$ |
| $s^3$ | $\cancel{0}\quad\epsilon$ | $\frac{7}{2}$ | $0$ |
| $s^2$ | $\frac{6\epsilon-7}{\epsilon}$ | $3$ | $0$ |
| $s^1$ | $\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$ | $0$ | $0$ |
| $s^0$ | $3$ | $0$ | $0$ |

# Example on Routh Stability

## Routh Table

| Label | First column | $\epsilon = +$ | $\epsilon = -$ |
|---|---|---|---|
| $s^5$ | $1$ | $+$ | $+$ |
| $s^4$ | $2$ | $+$ | $+$ |
| $s^3$ | $\cancel{0}$ $\epsilon$ | $+$ | $-$ |
| $s^2$ | $\dfrac{6\epsilon - 7}{\epsilon}$ | $-$ | $+$ |
| $s^1$ | $\dfrac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$ | $+$ | $+$ |
| $s^0$ | $3$ | $+$ | $+$ |

# Example on Routh Stability

## Interpretation of Results

**Two sign changes → two roots with positive real parts**

```
>> p=[1 2 3  6 5  3];
>> roots(p)

ans =

   0.3429 + 1.5083i
   0.3429 - 1.5083i
  -1.6681
  -0.5088 + 0.7020i
  -0.5088 - 0.7020i
```

# Example on Routh Stability

```
>> Routh_H([1 2  3 6  5  3])
-------------------------------------------------------------------------
                         RESULTS
-------------------------------------------------------------------------
  Not satisfied with the sufficient condition.
  The system is unstable and has 2 pole(s) in the RSP.
-------------------------------------------------------------------------


 ROUTH-HURWITZ ARRAY

  s^5 |     1     |     3     |     5     |
  s^4 |     2     |     6     |     3     |
  s^3 |     0     |    3.5    |
  s^2 |   -Inf    |     3     |
  s^1 |    3.5    |
  s^0 |     3     |
```

# 1 - Another Method

As the characteristics equation is

$$s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$$

Let, *s=1/d*

➡ $$\left(\frac{1}{d}\right)^n + a_{n-1}\left(\frac{1}{d}\right)^{n-1} + \ldots + a_1\left(\frac{1}{d}\right) + a_0 = 0$$

Then, *d* are reciprocal of the roots *s*

➡ $$\left(\frac{1}{d}\right)^n \left[1 + a_{n-1}d + \ldots + a_1 d^{n-1} + a_0 d^n\right] = 0$$

# Example on Routh Stability

**Determine the stability of the closed-loop transfer function**

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

➡ **Characteristics Eq$^{n}$:**

$$\boxed{3d^5 + 5d^4 + 6d^3 + 3d^2 + 2d + 1 = 0}$$

# Example on Routh Stability

| | | | |
|---|---|---|---|
| $d^5$ | 3 | 6 | 2 |
| $d^4$ | 5 | 3 | 1 |
| $d^3$ | 4.2 | 1.4 | |
| $d^2$ | 1.33 | 1 | |
| $d^1$ | $-1.75$ | | |
| $d^0$ | 1 | | |

**Two sign changes → two roots with positive real parts**

**No Zeros in first column**

# Routh-Hurwitz Stability Criterion

## Special Case 2

- An **all zero row** in the Routh array which corresponds to pairs of roots with opposite signs.

- Remedy:
  - form an **auxiliary polynomial** from the coefficients in the row above.
  - Replace the zero coefficients from the coefficients of the **differentiated auxiliary polynomial**.
  - If there is no sign change, the roots of the auxiliary equation define the roots of the system on the imaginary axis.

# Special Case 2 - Example

**For purely even or odd polynomials, as**

$$s^4 + 5s^2 + 6 = 0$$

**Row of zeros as:**

| | | | | |
|---|---|---|---|---|
| $s^4$ | 1 | 5 | 6 | 0 |
| $s^3$ | 0 | 0 | 0 | 0 |
| $s^2$ | | | | |
| $s^1$ | | | | |
| $s^0$ | | | | |

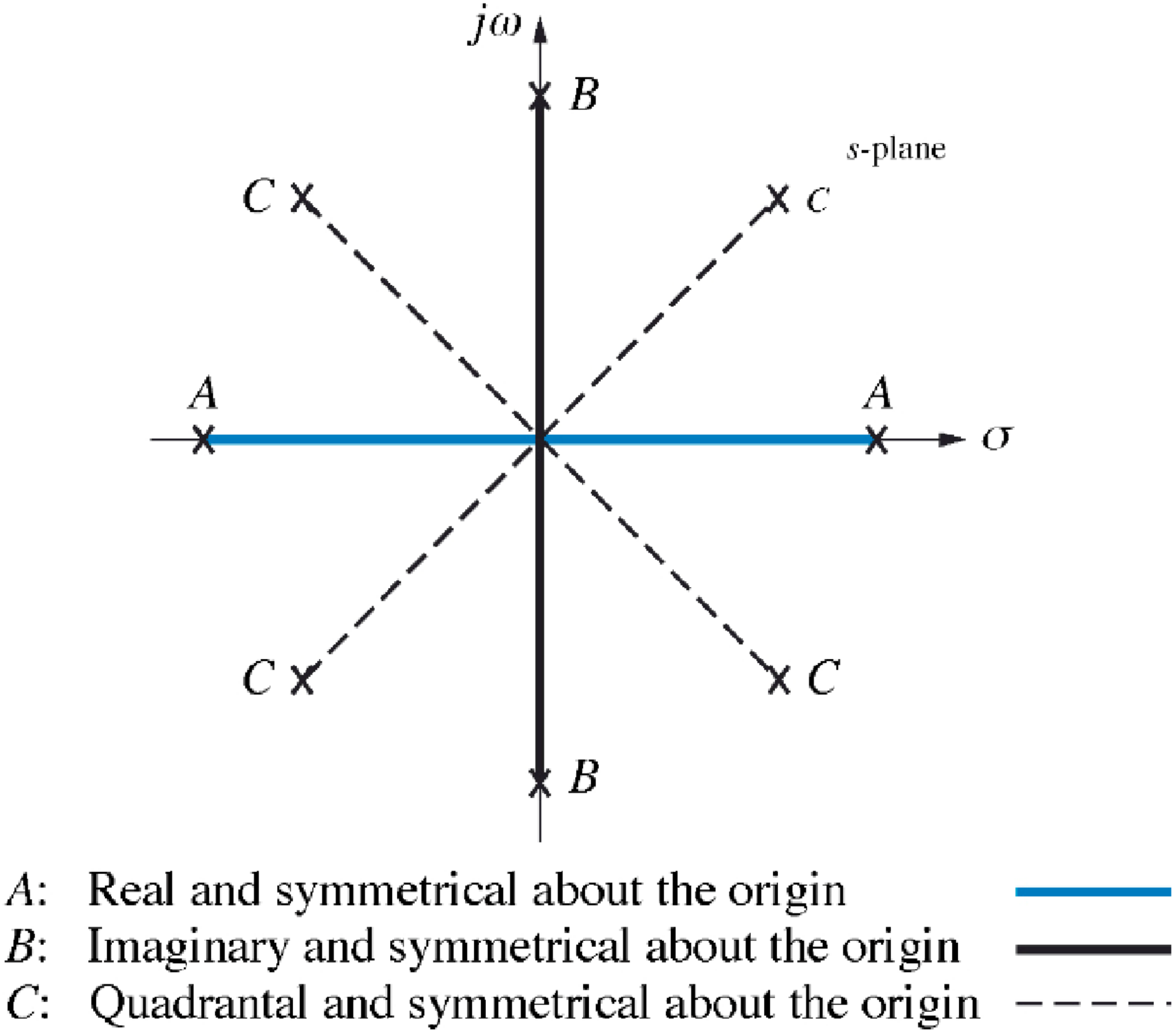
| | | | | |
|---|---|---|---|---|
| $s^4$ | 1 | 5 | 6 | 0 |
| $s^3$ | 4 | 10 | 0 | |
| $s^2$ | 2.5 | 6 | 0 | |
| $s^1$ | 0.4 | | | |
| $s^0$ | 6 | | | |

# Meaning of Entire Row of Zeros

```
>> p=[1 0 5 0  6];
>> roots(p)

ans =

      0 + 1.7321i
      0 - 1.7321i
      0 + 1.4142i
      0 - 1.4142i
```



*A*: Real and symmetrical about the origin

*B*: Imaginary and symmetrical about the origin

*C*: Quadrantal and symmetrical about the origin - - - -

# Meaning of Entire Row of Zeros

```
>> Routh_H([1  0  5  0 6])
------------------------------------------------------------------------------------
                         RESULTS
------------------------------------------------------------------------------------
  Not satisfied with the necessary condition.
  The system is critically stable and has 4 pole(s) in the imaginary axis .
  There were rows of zeroes in the array in the row(s) 2.
------------------------------------------------------------------------------------

 ROUTH-HURWITZ ARRAY

  s^4 |    1     |    5     |    6    |
  s^3 |    4     |    10    |
  s^2 |   2.5    |    6     |
  s^1 |   0.4    |
  s^0 |    6     |
```

# Routh-Hurwitz Stability Criterion

## Parameter Range Test

- The Routh-Hurwitz stability criterion may be used to find the **range of a parameter** for which the closed-loop systems is stable.

---

- Leave the parameter as an unknown coefficient in the characteristic polynomial, form the Routh array, check the range of the parameter such that the first column does not change sign.

# Routh-Hurwitz Stability Criterion

## Parameter Range Example

$$Q(s) = s^4 + 6s^3 + 11s^2 + 6s + K$$

| | | | | |
|---|---|---|---|---|
| $s^4$ | 1 | 11 | $K$ | 0 |
| $s^3$ | 6 | 6 | 0 | 0 |
| $s^2$ | 10 | $K$ | 0 | |
| $s^1$ | $c_1$ | 0 | | |
| $s^0$ | $d_1$ | | | |

$$c_1 = \frac{60 - 6K}{10} \qquad d_1 = K$$

Stability requires:

$$K > 0$$

$$60 - 6K > 0 \Rightarrow K < 10$$

$$\boxed{\therefore \quad 0 < K < 10}$$

# Routh-Hurwitz Stability Criterion

**Determine the stability of the closed-loop transfer function**

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

# Routh-Hurwitz Stability Criterion

| | | | |
|---|---|---|---|
| $s^5$ | 1 | 6 | 8 |
| $s^4$ | ~~7~~ 1 | ~~42~~ 6 | ~~56~~ 8 |
| $s^3$ | ~~0~~ ~~4~~ 1 | ~~0~~ ~~12~~ 3 | ~~0~~ ~~0~~ 0 |
| $s^2$ | 3 | 8 | 0 |
| $s^1$ | $\frac{1}{3}$ | 0 | 0 |
| $s^0$ | 8 | 0 | 0 |

$$P(s) = s^4 + 6s^2 + 8$$

$$\Downarrow$$

$$\frac{dP(s)}{ds} = 3s^3 + 12s + 0 \Rightarrow s^3 + 4s + 0$$

**Marginally Stable System**

# Routh-Hurwitz Stability Criterion

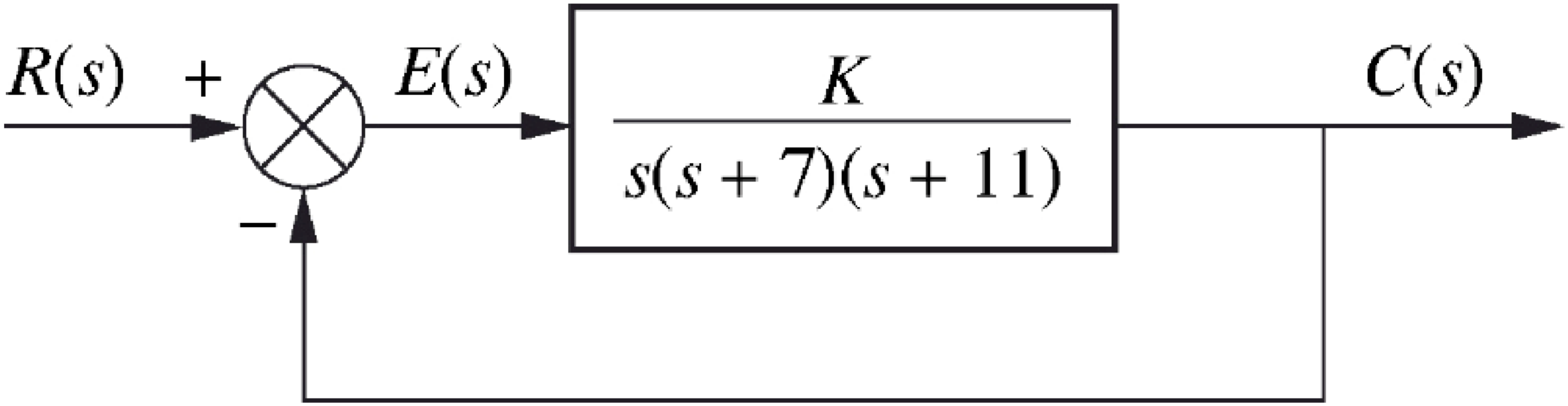
```
>> Routh_H([1 7 6 42 8 56])
---------------------------------------------------------------------------------------------------------------
                                   RESULTS
---------------------------------------------------------------------------------------------------------------
   The system is critically stable and has 4 pole(s) in the imaginary axis .
   There were rows of zeroes in the array in the row(s) 3.
---------------------------------------------------------------------------------------------------------------


 ROUTH-HURWITZ ARRAY

   s^5 |     1      |     6     |     8     |
   s^4 |     7      |    42     |    56     |
   s^3 |    28      |    84     |
   s^2 |    21      |    56     |
   s^1 |  9.3333    |
   s^0 |    56      |
```

# Routh-Hurwitz Stability Criterion

Find the range of $K$, for which the following system will be stable, marginally stable, & unstable. Assume $K>0$.



# Routh-Hurwitz Stability Criterion

| | | |
|---|---|---|
| $s^3$ | $1$ | $77$ |
| $s^2$ | $18$ | $K$ |
| $s^1$ | $\frac{1386-K}{18}$ | |
| $s^0$ | $K$ | |

$K<1386$ → Stable

$K>1386$ → Unstable

# Routh-Hurwitz Stability Criterion

$K=1386$

| | | |
|---|---|---|
| $s^3$ | 1 | 77 |
| $s^2$ | 18 | 1386 |
| $s^1$ | $\cancel{0}\quad 36$ | |
| $s^0$ | 1386 | |

Since there is no sign change → system is marginally stable

# Routh-Hurwitz Stability Criterion

$K$=1386

$$\frac{C}{R}=\frac{1386}{s^3+18s^2+77s+1386}$$

```
>> p=[1 18 77 1386];
>> roots(p)

ans =

 -18.0000
        0 + 8.7750i
        0  - 8.7750i
```

Since there is no sign change → system is marginally stable

# Routh-Hurwitz Stability Criterion

```
>> Routh_H([1  18 77 1386])
---------------------------------------------------------------------------
                        RESULTS
---------------------------------------------------------------------------
  The system is critically stable and has 2 pole(s) in the imaginary axis .
  There were rows of zeroes in the array in the row(s) 3.
---------------------------------------------------------------------------

 ROUTH-HURWITZ ARRAY

  s^3 |      1      |      77     |
  s^2 |     18      |     1386    |
  s^1 |     36      |
  s^0 |    1386     |
```

# Routh-Hurwitz Stability Criterion

```
>> Routh_H([1 -6  -7  -52])
--------------------------------------------------------------------------------
                              RESULTS
--------------------------------------------------------------------------------
   Not satisfied with the necessary condition.
   Not satisfied with the sufficient condition.
   The system is unstable and has 1 pole(s) in the RSP.
--------------------------------------------------------------------------------

 ROUTH-HURWITZ ARRAY

   s^3 |      1        |    -7    |
   s^2 |     -6        |   -52    |
   s^1 |  -15.6667     |
   s^0 |    -52        |
```

# System Response of $n^{th}$ Order

(i) First order system response

(ii) Second order system response

(iii) High order system response

# First order

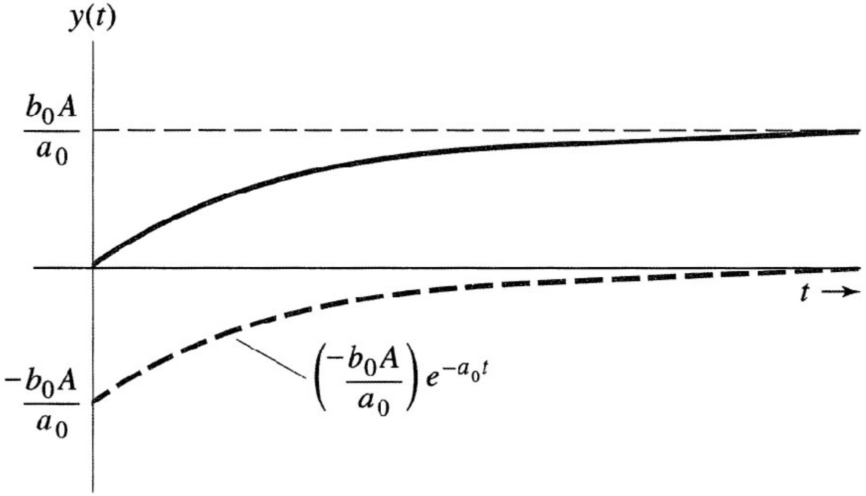
$$\frac{dy}{dt} + a_0 y = b_0 r$$

$$Y(s) = \frac{b_0}{s+a_0} R(s) + \frac{1}{s+a_0} y(0)$$

$$let \qquad r(t) = Au(t)$$

$$Y(s) = \frac{A\dfrac{b_0}{a_0}}{s} + \frac{-A\dfrac{b_0}{a_0}}{s+a_0} + \frac{1}{s+a_0} y(0)$$

$$y(t) = \frac{Ab_0}{a_0} u(t) - \frac{Ab_0}{a_0} e^{-a_0 t} + y(0) e^{-a_0 t}$$



# Second order

$$\frac{d^2y}{dt^2}+a_1\frac{dy}{dt}+a_0y=b_1\frac{dr}{dt}+b_0r$$

$$Y(s)=\frac{b_1s+b_0}{s^2+a_1s+a_0}R(s)+\frac{sy'(0)+(a_1+1)y(0)}{s^2+a_1s+a_0}$$

Three cases :

(a) Two characteristic roots are real and distinct.

(b) Two characteristic roots are equal.

(c) Two characteristic roots are complex numbers.

# Second order

Two characteristic roots are real and distinct.

$$\text{let} \qquad y'(0) = y(0) = 0 \qquad r(t) = u(t)$$

$$Y(s) = \frac{k_1}{s - s_1} + \frac{k_2}{s - s_2} + \frac{k_3}{s}$$

$$y(t) = (k_1 e^{s_1 t} + k_2 e^{s_2 t} + k_3) u(t)$$

# Second order

Two characteristic roots are equal

$$let \qquad y'(0) = y(0) = 0 \qquad r(t) = u(t)$$

$$Y(s) = \frac{k_1}{(s - s_1)^2} + \frac{k_2}{s - s_1} + \frac{k_3}{s}$$

$$y(t) = (k_1 e^{s_1 t} + k_2 t e^{s_1 t} + k_3) u(t)$$

# Second order

Two characteristic roots are complex numbers

$$let \qquad y'(0) = y(0) = 0 \qquad r(t) = u(t)$$

$$Y(s) = \frac{k_1}{(s+\sigma)^2 + \omega^2} R(s)$$

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} R(s)$$

$$\sigma = \xi\omega_n$$

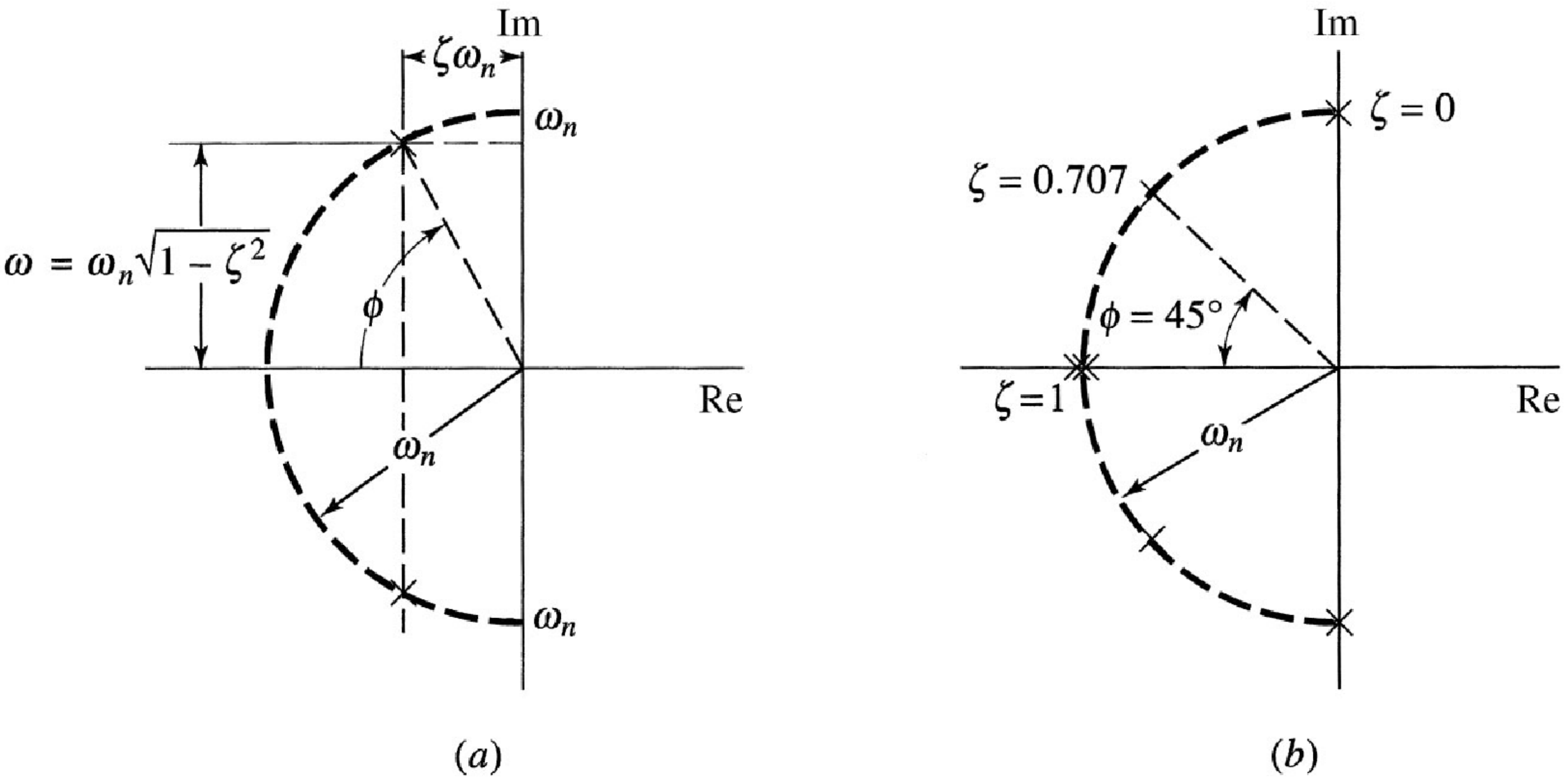
$$\omega = \omega_n\sqrt{1-\xi^2}$$

*Damping ratio*

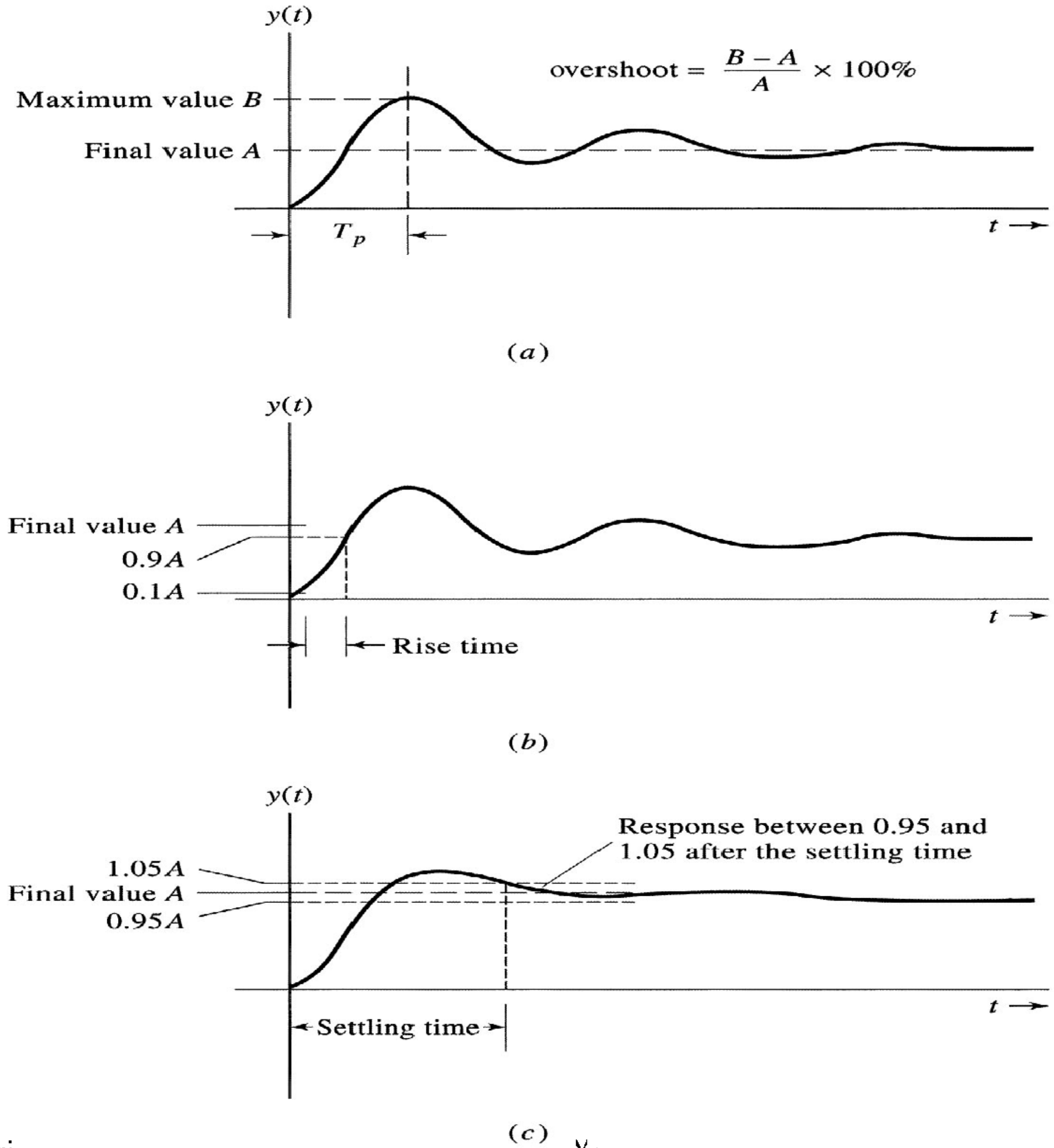
*Undamped natural frequency*

$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_n\sqrt{1-\xi^2}\,t + \cos^{-1}\xi)$$

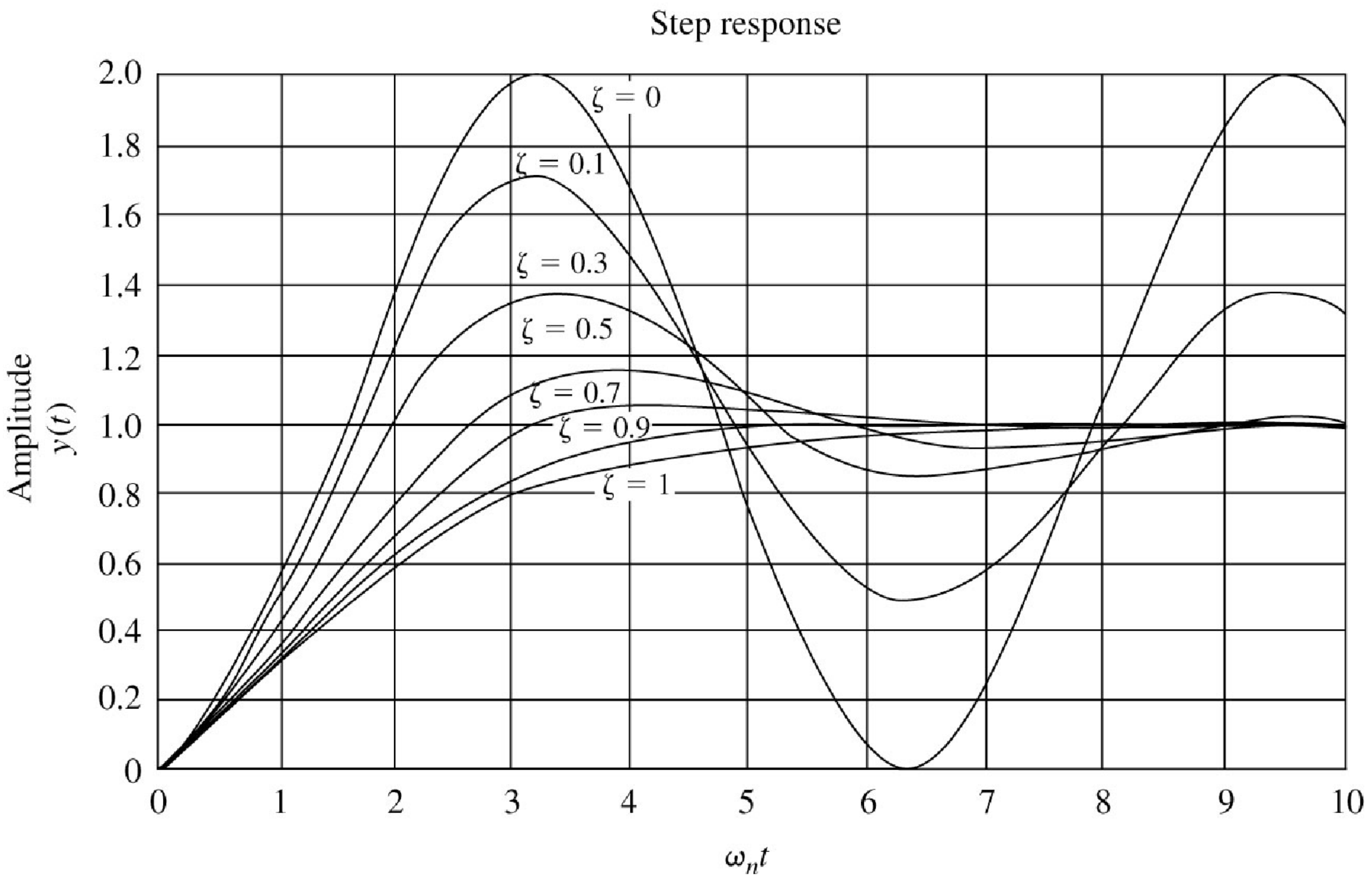
# Second order



# Second order



# Second order



# Higher-Order System

$$Y(s)=\frac{-8s^2+5}{s^4+9s^3+37s^2+81s+52}$$

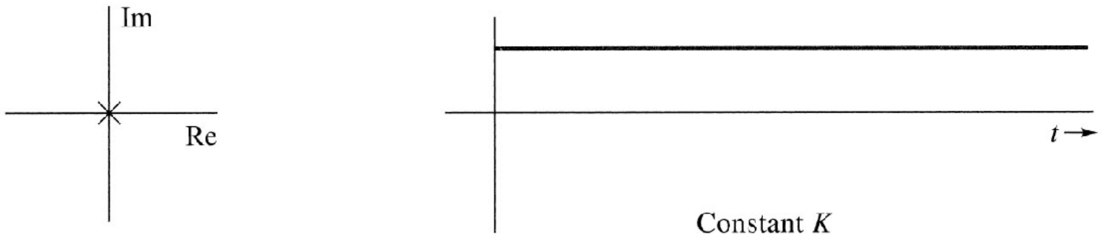
$$Y(s)=\frac{k_1}{s+1}+\frac{k_2}{s+4}+\frac{k_3s+k_4}{s^2+4s+13}$$

*Dominant root*

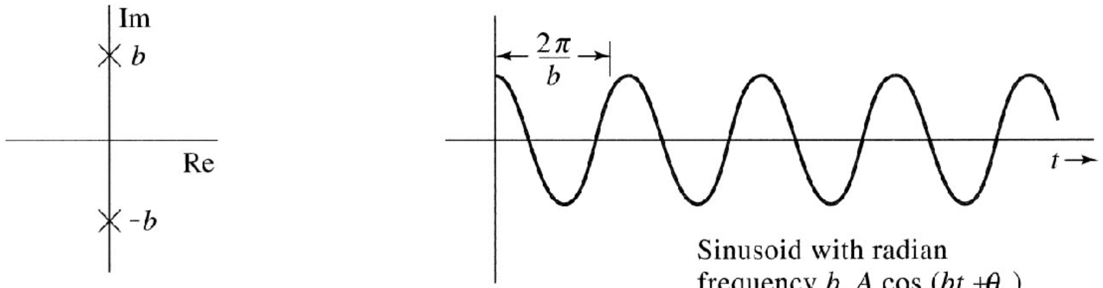
*nondominant root*

**Denominator Root Locations**

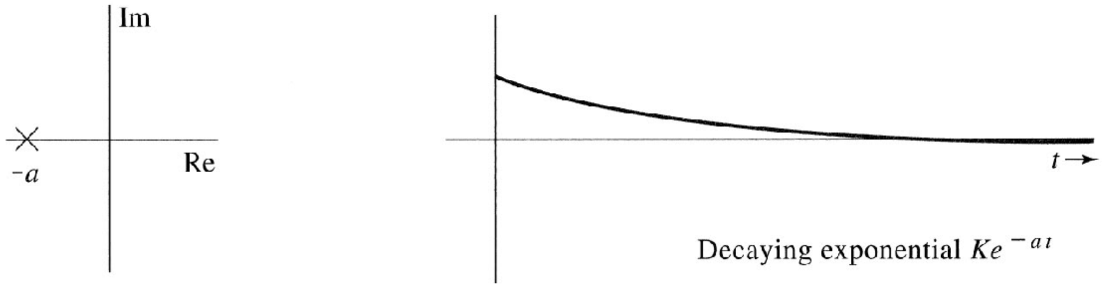
**Time Function**



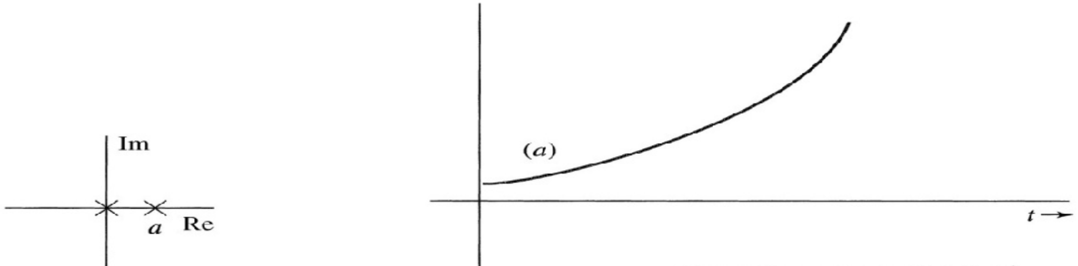
(*a*)



(*b*)

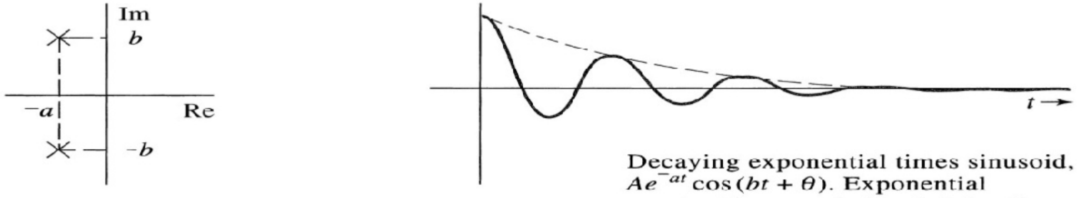


(*c*)



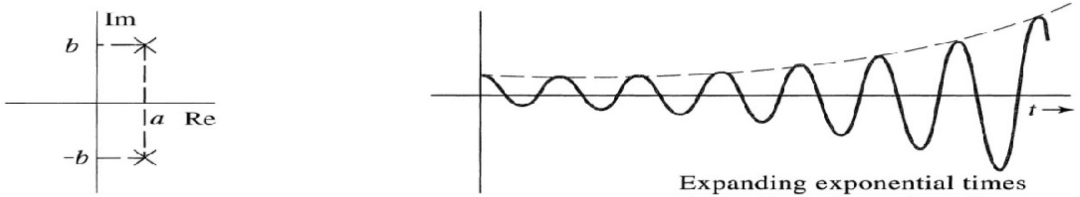
Expanding exponential $Ke^{at}$

(d)



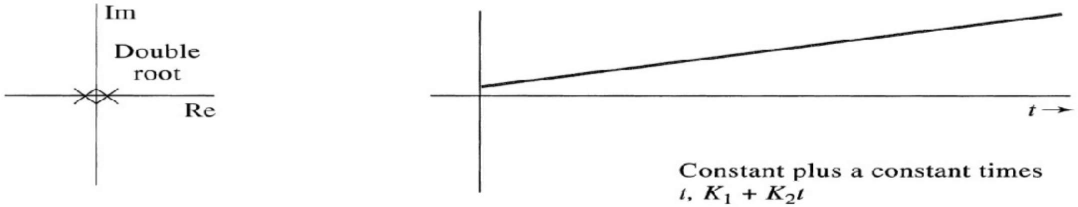
Decaying exponential times sinusoid, $Ae^{-at}\cos(bt + \theta)$. Exponential constant is $-a$ and sinusiodal radian frequency is $b$.

(e)



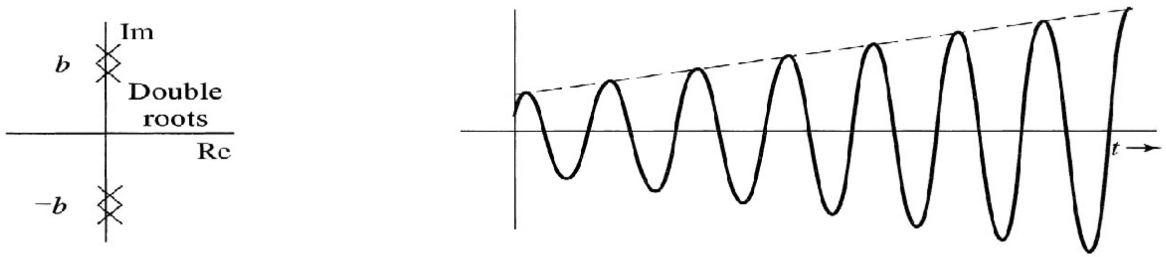
Expanding exponential times sinusoid, $Ae^{at}\cos(bt + \theta)$.

(f)



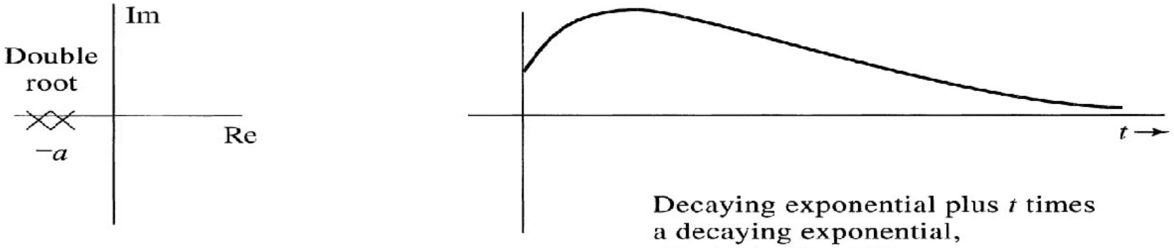
Constant plus a constant times $t$, $K_1 + K_2t$

(g)



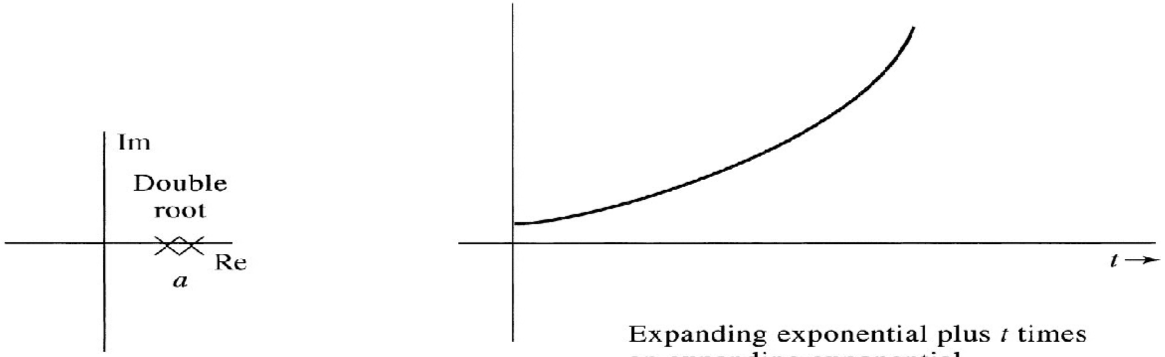
Sinusoid plus *t* times a sinusiod,
$A_1 \cos(bt + \theta_1) + A_2 t \cos(bt + \theta_2)$

(*h*)



Decaying exponential plus *t* times
a decaying exponential,
$K_1 e^{-at} + K_2 t e^{-at}$

(*i*)



Expanding exponential plus *t* times
an expanding exponential,
$K_1 e^{at} + K_2 t e^{at}$

(*j*)

# END