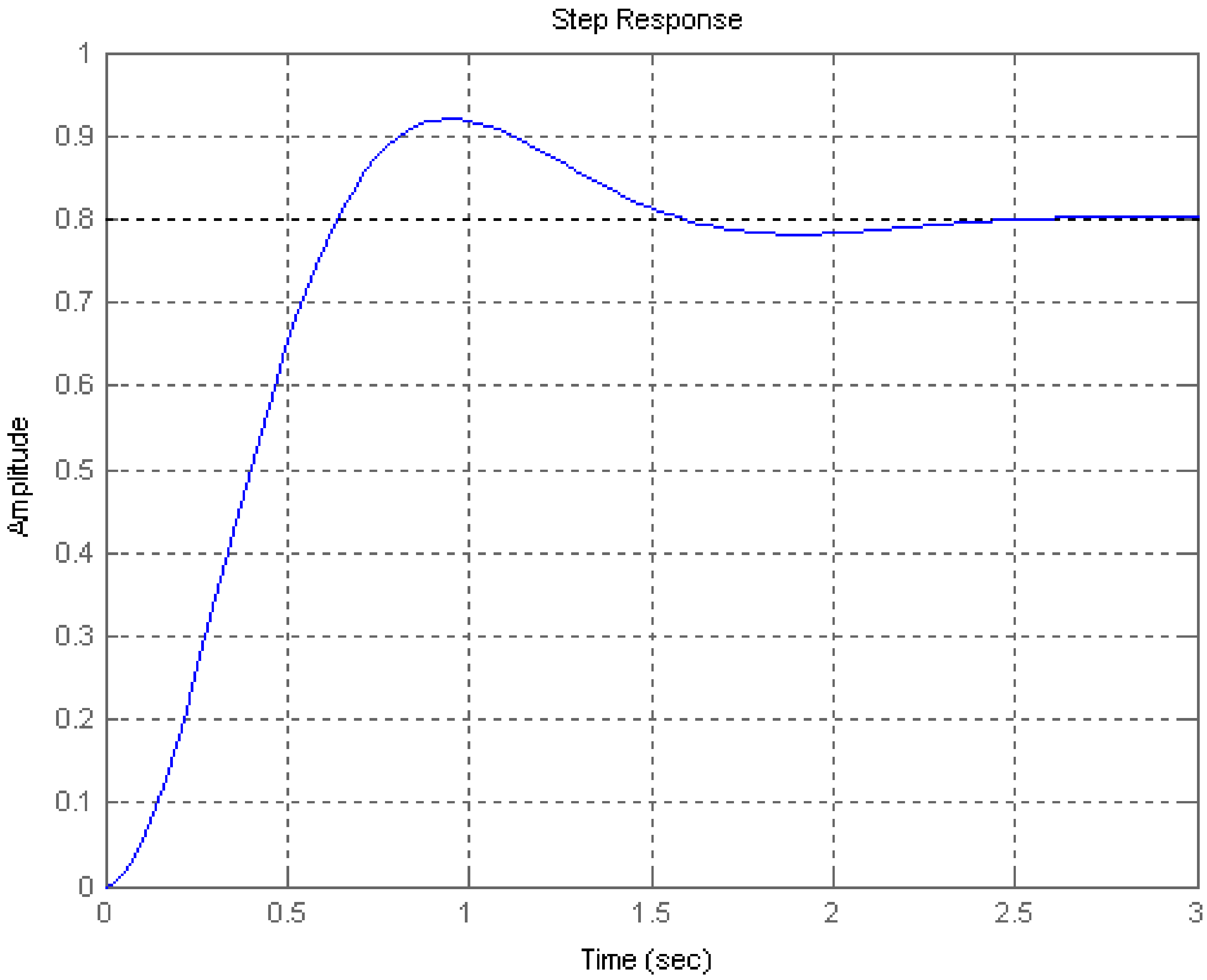
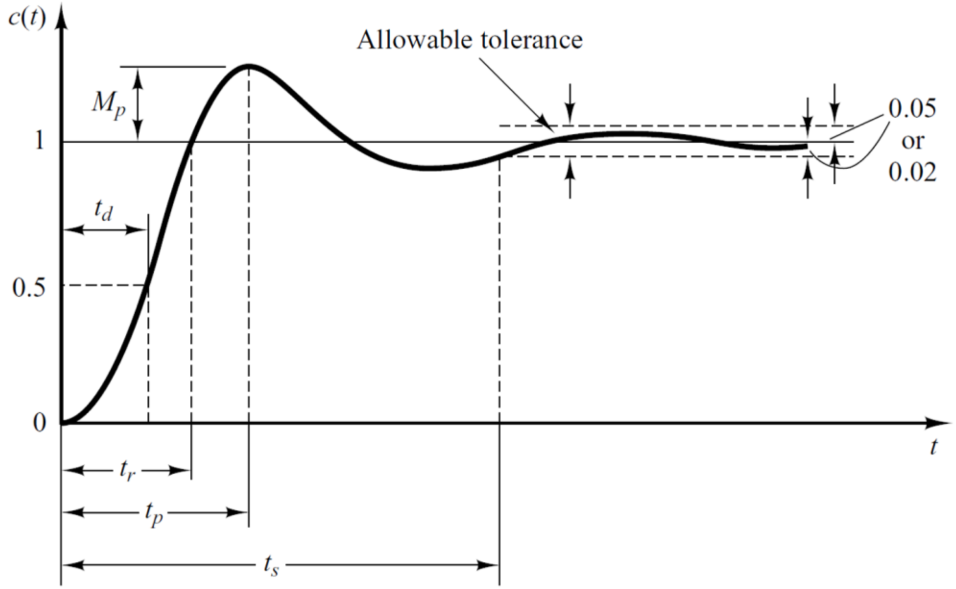
System Types & Error Constants

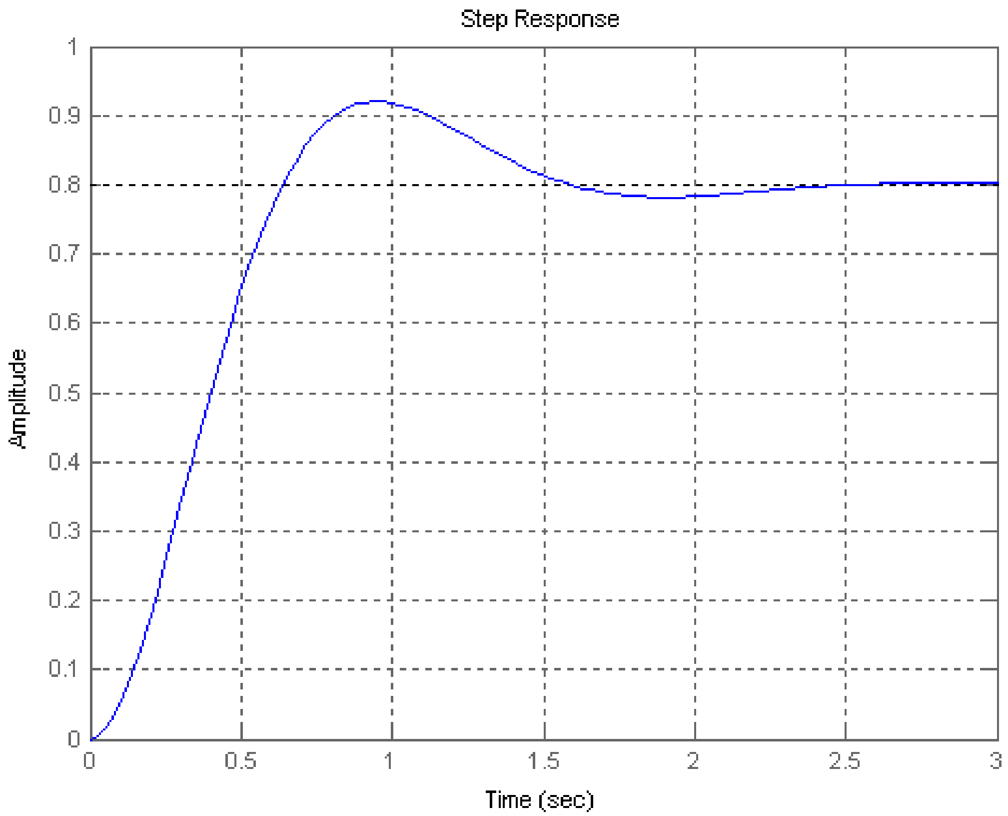
MARK the definitions and then CALCULTAE the numerical values of the following specifications:

a) Delay time, t_d
b) Rise time, t_r
c) Peak time, t_p
d) Settling time, t_s for 2%
e) Maximum overshoot, M_p
f) Steady state response. y_{ss}
g) Steady state error, e_{ss}
h) Maximum amplitude, y_{max}
i) Percentage overshoot
j) Natural Frequency, ω_n
k) Damped Natural Frequency, ω_d
l) Damping Ratio, ζ





$$t_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} \qquad t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$



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k) Damped Natural Frequency, ω_d
l) Damping Ratio, ζ

$$\sigma = \zeta\omega_n$$

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma}$$

$$t_r = \frac{\pi - \beta}{\omega_d}$$

$$t_s = \frac{4}{\zeta\omega_n} \quad \text{(2\% criterion)}$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right\}$$

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$e_{ss} = \lim_{t\to\infty}[r(t) - c(t)]$$

$$\% MP = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$$

Introduction

- ***Errors in a control system*** can be attributed to many factors:
  - **Imperfections in the system components**
    (*e.g.* static friction, amplifier drift, aging, deterioration, *etc*…)
  - **Changes in the reference inputs** $\rightarrow$
    cause errors during transient periods & may cause steady-state errors.

- In this section, we shall investigate a **type of steady-state error that is caused by the incapability of a system to follow particular types of inputs.**

Steady-State Errors with Respect to Types of Inputs

- Any physical control system inherently suffers steady-state response to certain types of inputs.

- A system may have no steady-state error to a step input, but the same system exhibit nonzero steady-state error to a ramp input.

- Whether a given unity feedback system will exhibit steady-state error for a given type of input depends on the type of ***loop gain*** of the system.

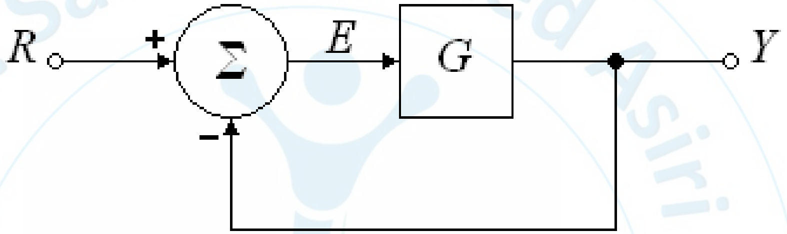
Classification of Control System

- Control systems may be ***classified according to their ability to track polynomial inputs***, or *in other words*, their ability to reach zero steady-state to step-inputs, ramp inputs, parabolic inputs and so on.

- This is a reasonable classification scheme because actual inputs may frequently be considered combinations of such inputs.

- The magnitude of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

The Unity Feedback Control Case
Steady-State Error



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

- Error: $e(t) = r(t) - y(t) \Rightarrow E(s) = R(s) - Y(s) = \dfrac{R(s)}{1+G(s)}$

- *Using the FVT, the **steady-state error** is given by:*

$$e_{ss} = \lim_{t\to\infty} e\,(t) = \lim_{s\to 0}\; sE(s) = \lim_{s\to 0}\; sR(s)\frac{1}{1+G(s)}$$

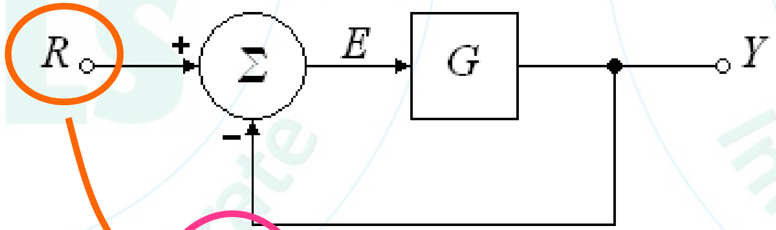
FVT

Steady-state error to polynomial input - Unity Feedback Control -

- Consider a ***polynomial input***:

$$r(t) = t^{k-1}u(t) \Rightarrow R(s) = \frac{1}{s^k}$$

- The ***steady-state error*** is then given by:



$$e_{ss} = \lim_{s\to 0} s \frac{1}{s^k}\frac{1}{1+G(s)} = \lim_{s\to 0} \frac{s}{s^k}\frac{1}{1+G(s)}$$

System Type

A *unity feedback system* is defined to be **type k** if the feedback system guarantees:

$$e_{ss} = 0 \quad for \quad R(s) = \frac{1}{s^k}$$

$$|e_{ss}| < \infty \quad for \quad R(s) = \frac{1}{s^{k+1}}$$

System Type (cont'd)

- Since, for an input

$$R(s) = \frac{1}{s^k}$$

$$e_{ss} = \lim_{s\to 0} s \frac{1}{s^k} \frac{1}{1+G(s)} = \lim_{s\to 0} \frac{s}{s^k} \frac{1}{1+G(s)}$$

the *system* is called a **type k system** if:

$$\lim_{s\to 0} \frac{s}{s^k} \frac{1}{1+G(s)} = 0$$

$$\left|\lim_{s\to 0} \frac{s}{s^{k+1}} \frac{1}{1+G(s)}\right| < \infty$$

Example 1: Unity feedback

- Given a stable system whose the open-loop transfer function is:

$$G(s) = \frac{K(s-z_1)(s-z_2)\cdots}{s\ (s-p_1)(s-p_2)\cdots} = \frac{G_0(s)}{s} \text{ subjected to inputs } \quad R(s) = \frac{1}{s^k}$$

$(p_i \neq 0)$

- Step function: $R(s) = 1/s\,,\ k = 1$

$$e_{ss} = \lim_{s\to 0} \frac{s}{s} \frac{1}{1+\frac{G_0(s)}{s}} = \lim_{s\to 0} \frac{s}{s} \frac{s}{s+G_0(s)} = \frac{0}{0+G_0(0)} = 0$$

- Ramp function: $R(s) = 1/s^2\,,\ k = 2$

$$e_{ss} = \lim_{s\to 0} \frac{s}{s^2} \frac{1}{1+\frac{G_0(s)}{s}} = \lim_{s\to 0} \frac{s}{s^2} \frac{s}{s+G_0(s)} = \lim_{s\to 0} \frac{1}{s+G_0(s)} = \frac{1}{G_0(0)} \neq 0$$

→ The system is type 1

Example 2: Unity feedback

- Given a stable system whose the open-loop transfer function is:

$$G(s) = \frac{K(s-z_1)(s-z_2)\cdots}{s^2(s-p_1)(s-p_2)\cdots} = \frac{G_0(s)}{s^2}$$ subjected to inputs $$R(s) = \frac{1}{s^k}$$

$(p_i \neq 0)$

- Step function: $R(s) = 1/s\,,\ k = 1$

$$e_{ss} = \lim_{s\to 0} \frac{s}{s} \frac{1}{1+\frac{G_0(s)}{s^2}} = \lim_{s\to 0} \frac{s}{s} \frac{s^2}{s^2 + G_0(s)} = \frac{0}{0+G_0(0)} = 0$$

- Ramp function: $R(s) = 1/s^2\,,\ k = 2$

$$e_{ss} = \lim_{s\to 0} \frac{s}{s^2} \frac{1}{1+\frac{G_0(s)}{s^2}} = \lim_{s\to 0} \frac{s}{s^2} \frac{s^2}{s^2 + G_0(s)} = \lim_{s\to 0} \frac{s}{s^2 + G_0(s)} = \frac{0}{G_0(0)} = 0$$

- Parabola function: $R(s) = 1/s^3\,,\ k = 3$

$$e_{ss} = \lim_{s\to 0} \frac{s}{s^3} \frac{1}{1+\frac{G_0(s)}{s^2}} = \lim_{s\to 0} \frac{s}{s^3} \frac{s^2}{s^2 + G_0(s)} = \frac{1}{G_0(0)} \neq 0$$

→ type 2

Example 3: Unity feedback

- Given a stable system whose the open loop transfer function is:

$$G(s) = \frac{K(s - z_1)(s - z_2)\cdots}{(s - p_1)(s - p_2)\cdots} = G_0(s) \quad \text{subjected to inputs} \qquad R(s) = \frac{1}{s^k}$$

$$(p_i \neq 0)$$

- Step function: $R(s) = 1/s\,,\ k = 1$

$$e_{ss} = \lim_{s\to 0} \frac{s}{s}\frac{1}{1 + G_0(s)} = \frac{1}{1 + G_0(0)} \neq 0$$

→ The system is type 0

- Impulse function: $R(s) = 1,\ k = 0$

$$e_{ss} = \lim_{s\to 0} \frac{s}{1}\frac{1}{1 + G_0(s)} = \frac{0}{G_0(0)} = 0$$

Summary – Unity Feedback

- Assuming $p_i \neq 0$, unity system loop transfers such as:

$$G(s) = \frac{K(s-z_1)(s-z_2)\cdots}{(s-p_1)(s-p_2)\cdots} = G_0(s) \quad \rightarrow \text{type } 0$$

$$G(s) = \frac{K(s-z_1)(s-z_2)\cdots}{s\ (s-p_1)(s-p_2)\cdots} = \frac{G_0(s)}{s} \quad \rightarrow \text{type } 1$$

$$G(s) = \frac{K(s-z_1)(s-z_2)\cdots}{s^2(s-p_1)(s-p_2)\cdots} = \frac{G_0(s)}{s^2} \quad \rightarrow \text{type } 2$$

$$G(s) = \frac{K(s-z_1)(s-z_2)\cdots}{s^n(s-p_1)(s-p_2)\cdots} = \frac{G_0(s)}{s^n} \quad \rightarrow \text{type } n$$

General Rule – Unity Feedback

- An unity feedback system is of ***type k*** if the open-loop transfer function of the system has:

k poles at s=0

In other words,

- An unity feedback system is of ***type k*** if the open-loop transfer function of the system has:

k integrators

Error Constants

- A *stable* ***unity*** *feedback system is* ***type k*** with respect to reference inputs if the **open loop transfer function has k poles at the origin**:

$$G(s) = \frac{(s - z_1)(s - z_2)\cdots}{s^k (s - p_1)(s - p_2)\cdots} = \frac{G_0(s)}{s^k}$$

Then the **error constant** is given by:

$$K_k = \lim_{s \to 0} s^k \, G(s) = G_0(0)$$

- The higher the constants, the smaller the steady-state error.

Error Constants

- For a ***Type 0 System***, the error constant, called ***position constant***, is given by:

$$K_p = \lim_{s \to 0} G(s) \qquad \text{(dimensionless)}$$

- For a ***Type 1 System***, the error constant, called ***velocity constant***, is given by:

$$K_v = \lim_{s \to 0} s\, G(s) \qquad (\text{sec}^{-1})$$

- For a ***Type 2 System***, the error constant, called ***acceleration constant,*** is given by:

$$K_a = \lim_{s \to 0} s^2\, G(s) \qquad (\text{sec}^{-2})$$

Steady-State Errors as a function of System Type – Unity Feedback

System type	Step input	Ramp input	Parabola input
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

Error Constants

Example:

- A temperature control system is found to have zero error to a constant tracking input and an error of 0.5°C to a tracking input that is linear in time, rising at the rate of 40°C/sec.
- What is the system type?

The system is type 1

- What is the steady-state error?

$$e_{ss} = 0.5^oC = \frac{40^oC/\text{sec}}{K_v}$$

- What is the error constant?

$$K_v = \frac{40^oC/\text{sec}}{0.5^oC\ \text{sec}^{-1}}$$

Error Constants

Conclusion

- *Classifying a system as **k type*** indicates the ability of the system to **achieve zero steady-state error to polynomials r(t) of degree less but not equal to k.**

- The system is ***type k*** if the **error is zero to all polynomials r(t) of degree less than k but non-zero for a polynomial of degree k.**

Error Constants

Conclusion

- A *stable* ***unity*** *feedback system is* ***type k*** with respect to reference inputs if the **loop transfer function has k poles at the origin**:

$$G(s) = \frac{(s - z_1)(s - z_2)\cdots}{s^k(s - p_1)(s - p_2)\cdots}$$

$$K_k = \lim_{s \to 0} s^k\, G(s)$$

- Then the **error constant** is given by:

The Classical Three- Term Controllers

Basic Operations of a Feedback Control

Think of what goes on in domestic hot water thermostat:

- The temperature of the water is measured.
- Comparison of the measured and the required values provides an error, *e.g. "too hot' or 'too cold'.*
- On the basis of error, a control algorithm decides what to do.

  $\rightarrow$ *Such an algorithm might be:*

  – *If the temperature is too high then turn the heater off.*

  – *If it is too low then turn the heater on*
- The adjustment chosen by the control algorithm is applied to some adjustable variable, *such as the power input to the water heater.*

Feedback Control Properties

- ***A feedback control system seeks to bring the measured quantity to its required value or set-point.***
- The control system *does not need to know why the measured value is not currently what is required*, only that is so.
- There are two *possible causes of such a disparity*:
  - The system has been disturbed.
  - The setpoint has changed. In the absence of external disturbance, a change in setpoint will introduce an error. The control system will act until the measured quantity reach its new setpoint.

The PID Algorithm

- The **PID algorithm** is the *most popular feedback controller algorithm used*. It is a robust easily understood algorithm that can provide *excellent control performance* despite the varied dynamic characteristics of processes.

- As the name suggests, the PID **algorithm** consists of ***three basic modes***:

  the ***Proportional*** mode,

  the ***Integral*** mode

  & the ***Derivative*** mode.

P, PI or PID Controller

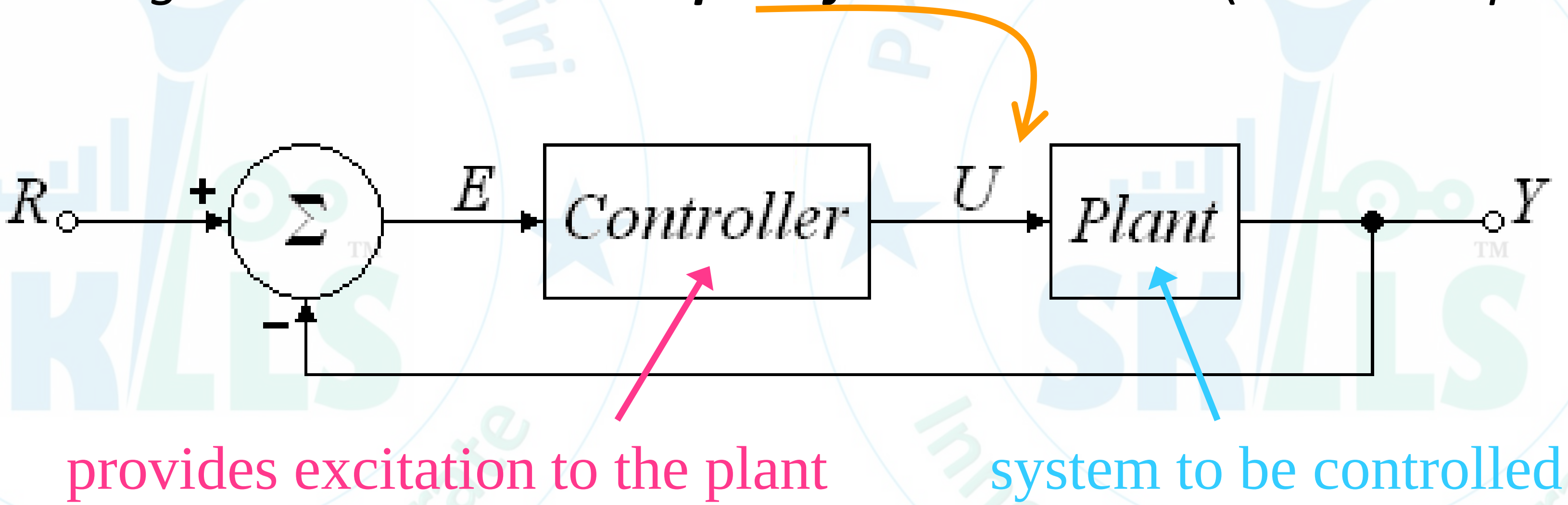
- When utilizing the PID algorithm, it is necessary to decide which modes are to be used (P, I or D) and then ***specify the parameters (or settings) for each mode used***.
- Generally, three basic algorithms are used: ***P, PI or PID***.
- Controllers are designed to eliminate the need for continuous operator attention.

→ *Cruise control in a car and a house thermostat*

are common examples of *how controllers are used to automatically adjust some variable to hold a measurement (or* ***process variable****) to a desired variable (or* ***set-point****)*

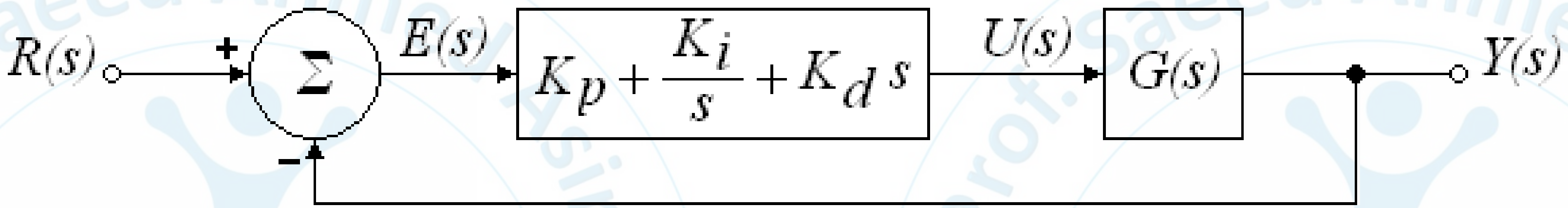
Controller Output

- The *variable being controlled* is the ***output of the controller*** (and the *input of the plant*):



- The output of the controller will change in response to a change in measurement or set-point (that said a change in the tracking error)

PID Controller



- In the *s*-domain, the PID controller may be represented as:

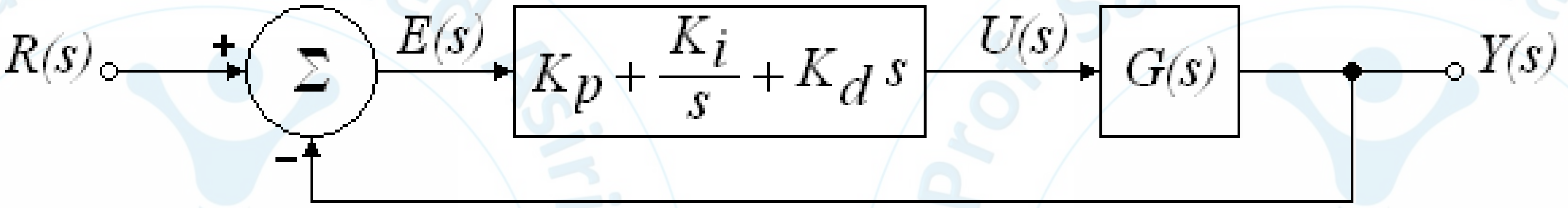
$$U(s) = \left(K_p + \frac{K_i}{s} + K_d s\right) E(s)$$

- In the time domain:

$$u(t) = K_p e(t) + K_i \int_0^t e(t)dt + K_d \frac{de(t)}{dt}$$

proportional gain | integral gain | derivative gain

PID Controller



- In the time domain:

$$u(t) = K_p e(t) + K_i \int_0^t e(t)dt + K_d \frac{de(t)}{dt}$$

- The signal *u(t)* will be sent to the plant, and a new output *y(t)* will be obtained. This new output *y(t)* will be sent back to the sensor again to find the new error signal *e(t).* The controllers takes this new error signal and computes its derivative and its integral gain. This process goes on and on.

PID Controller

Definitions

- In the time domain:

$$u(t) = K_p e(t) + K_i \int_0^t e(t)dt + K_d \frac{de(t)}{dt}$$

$$= K_p \left(e(t) + \frac{1}{T_i} \int_0^t e(t)dt + T_d \frac{de(t)}{dt} \right)$$

integral time constant

derivative time constant

$$where \quad T_i = \frac{K_p}{K_i}, \qquad T_d = \frac{K_d}{K_i}$$

derivative gain

proportional gain

integral gain

PID Controller

Controller Effects

- A **proportional controller** (P) ***reduces error responses to disturbances***, but *still allows a steady-state error.*

- When the controller includes a term proportional to the integral of the error (I), then the ***steady state error to a constant input is eliminated***, although typically *at the cost of deterioration in the dynamic response.*

- A **derivative control** typically ***makes the system better damped and more stable.***

PID Controller

Closed-loop Response

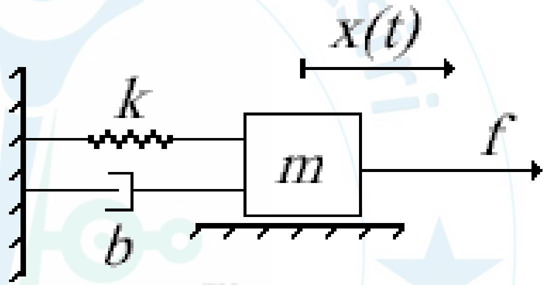
	Rise time	Maximum overshoot	Settling time	Steady-state error
P	**Decrease**	Increase	Small change	**Decrease**
I	**Decrease**	Increase	Increase	**Eliminate**
D	Small change	**Decrease**	**Decrease**	Small change

- Note that these correlations may not be exactly accurate, because P, I and D gains are dependent of each other.

Open Loop System

Example:

- Suppose we have a simple mass, spring, damper problem.



- The dynamic model is such as:

$$m\ddot{x} + b\dot{x} + kx = f$$

- Taking the Laplace Transform, we obtain:

$$ms^2X(s) + bsX(s) + kX(s) = F(s)$$

- The Transfer function is then given by:

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

Open Loop System

Example:

- Let

$$m = 1kg, \quad b = 10N.s/m, \quad k = 20N/m, \quad f = 1N$$

- By plugging these values in the transfer function:

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 10s + 20}$$

- The goal of this problem is to show you how each of K_p, K_i *and* K_d contribute to obtain:

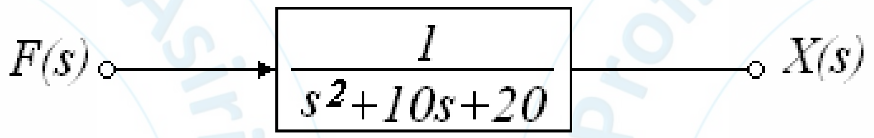
fast rise time,

minimum overshoot,

no steady-state error.

Open Loop System

Example:



- The (***open***) loop transfer function is given by:

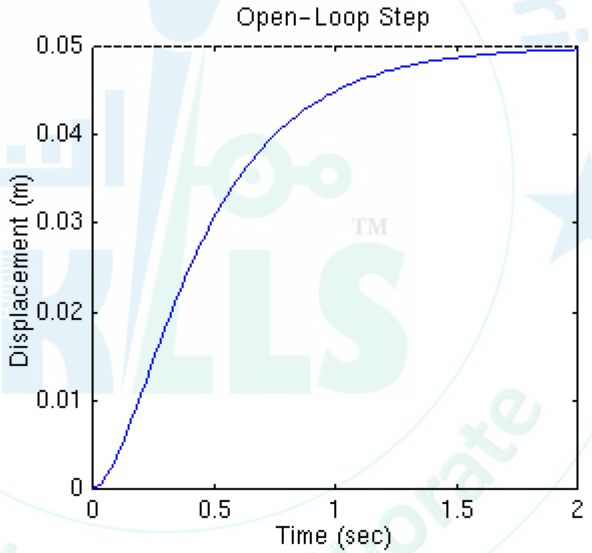
$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 10s + 20}$$

- The steady-state value for the output is:

$$x_{ss} = \lim_{t\to\infty} x\,(t) = \lim_{s\to 0} s\,X(s) = \lim_{s\to 0} s\,F(s)\,\frac{X(s)}{F(s)} = \frac{1}{20}$$

Open Loop System

Example:

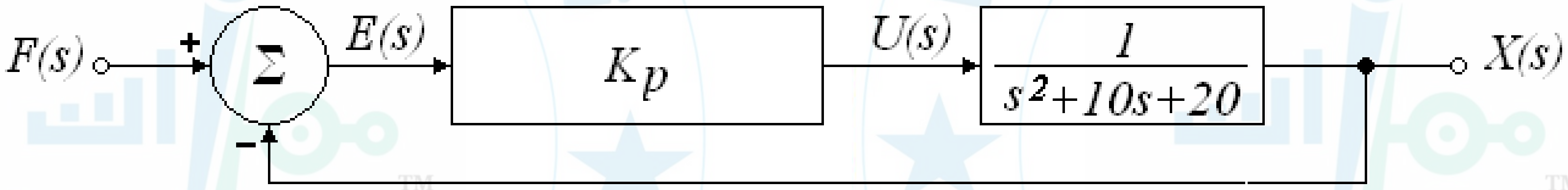


- 1/20=0.05 is the *final value* of the output to an *unit* step input.
- This corresponds to a **steady-state error of 95%, *quite large!***
- The ***settling time is about 1.5 sec.***

Closed Loop System

Example:

Proportional Controller



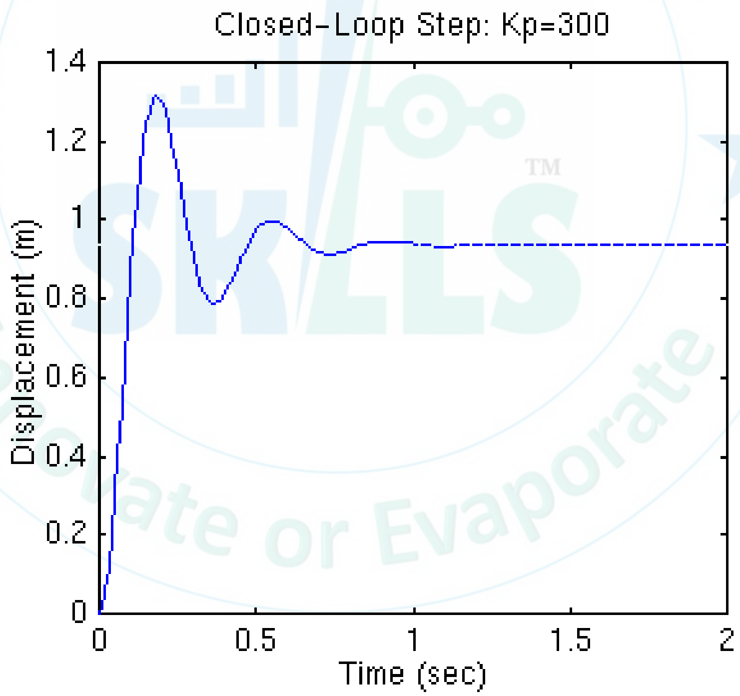
- The closed loop transfer function is given by:

$$\frac{X(s)}{F(s)} = \frac{\frac{K_p}{s^2+10s+20}}{1+\frac{K_p}{s^2+10s+20}} = \frac{K_p}{s^2+10s+(20+K_p)}$$

Closed Loop System

Example:

Proportional Controller

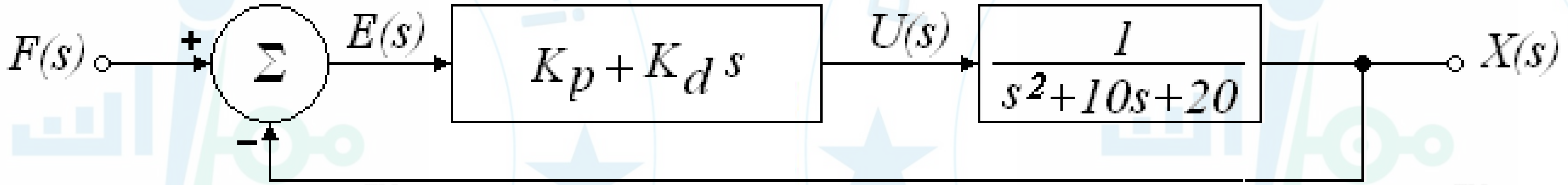


- Let $K_p = 300$
- The above plot shows that the ***proportional controller reduced both the rise time and the steady-state error***, *increased the overshoot*, and *decreased the settling time by small amount.*

Closed Loop System

Example:

PD Controller



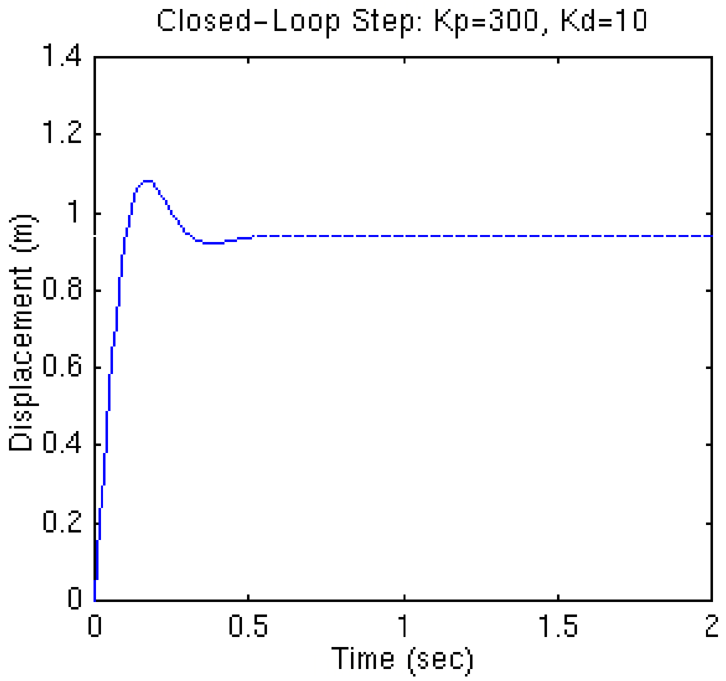
- The closed loop transfer function is given by:

$$\frac{X(s)}{F(s)} = \frac{\dfrac{K_p + K_d s}{s^2 + 10s + 20}}{1 + \dfrac{K_p + K_d s}{s^2 + 10s + 20}} = \frac{K_p + K_d s}{s^2 + (10 + K_d)s + (20 + K_p)}$$

Closed Loop System

Example:

PD Controller

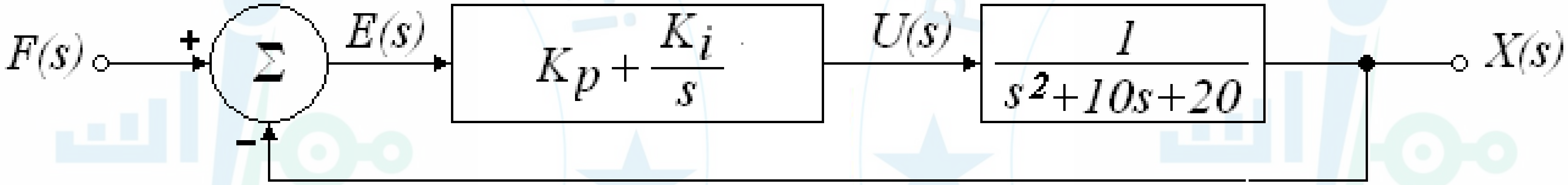


- Let $K_p = 300, \quad K_d = 10$
- This plot shows that the ***proportional derivative controller reduced both the overshoot and the settling time***, and *had small effect on the rise time and the steady-state error*.

Closed Loop System

Example:

PI Controller



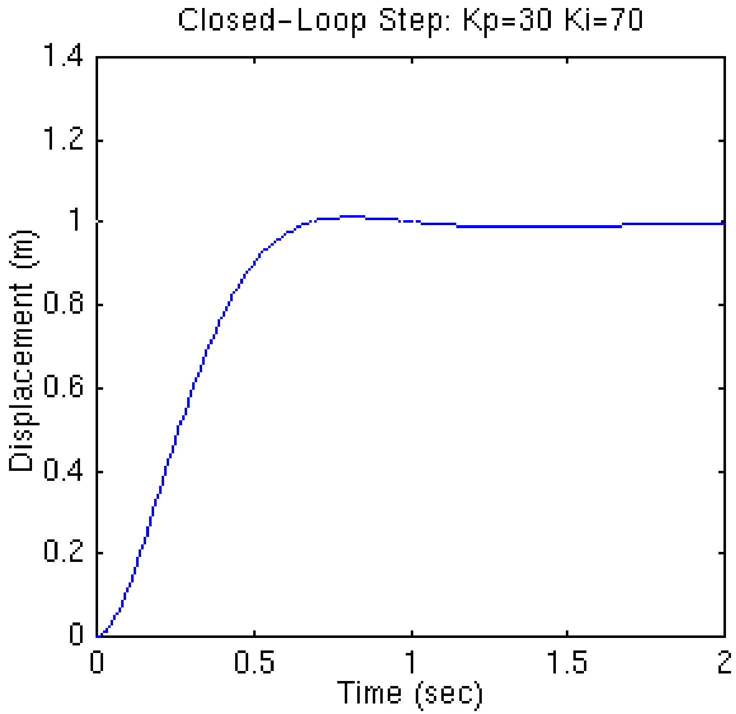
- The closed loop transfer function is given by:

$$\frac{X(s)}{F(s)} = \frac{\dfrac{K_p + K_i/s}{s^2 + 10s + 20}}{1 + \dfrac{K_p + K_i/s}{s^2 + 10s + 20}} = \frac{K_p s + K_i}{s^3 + 10s^2 + (20 + K_p)s + K_i}$$

Closed Loop System

Example:

PI Controller

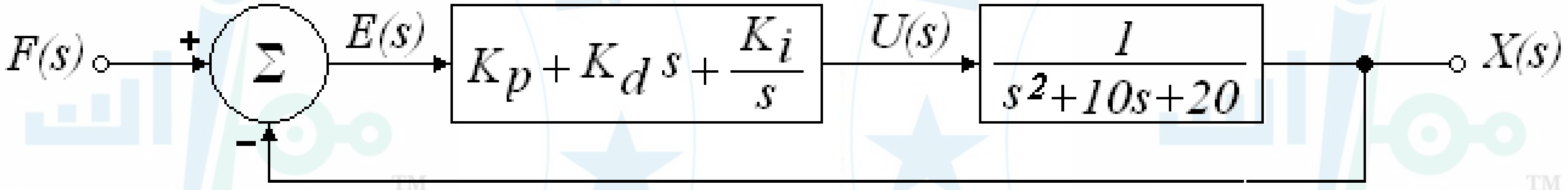


- Let $K_p = 30, \quad K_i = 70$
- We have reduced the proportional gain because the integral controller also ***reduces the rise time and increases the overshoot*** as the proportional controller does (double effect).
- The above response shows that the ***integral controller eliminated the steady-state error***.

Closed Loop System

Example:

PID Controller



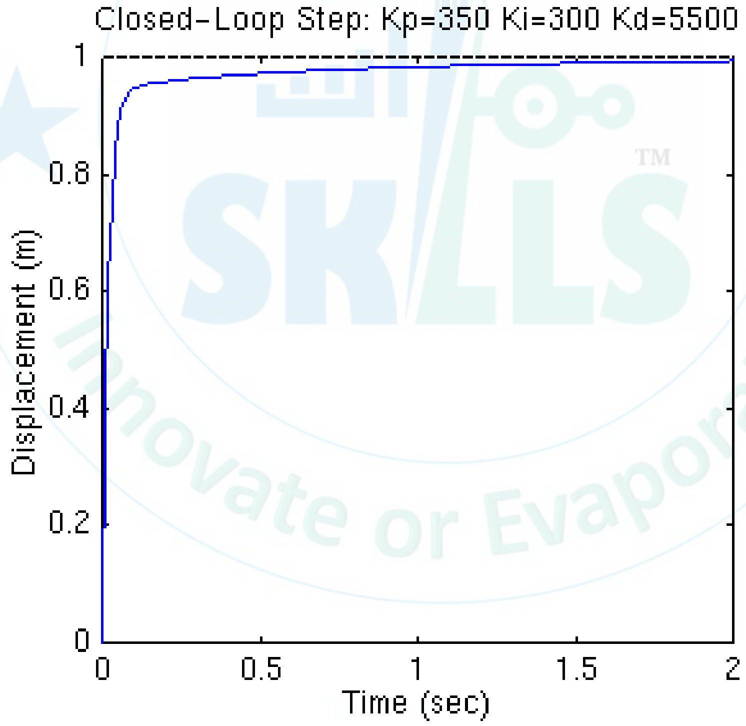
- The closed loop transfer function is given by:

$$\frac{X(s)}{F(s)} = \frac{\frac{K_p + K_d s + K_i/s}{s^2 + 10s + 20}}{1 + \frac{K_p + K_d s + K_i/s}{s^2 + 10s + 20}} = \frac{K_d s^2 + K_p s + K_i}{s^3 + (10 + K_d)s^2 + (20 + K_p)s + K_i}$$

Closed Loop System

Example:

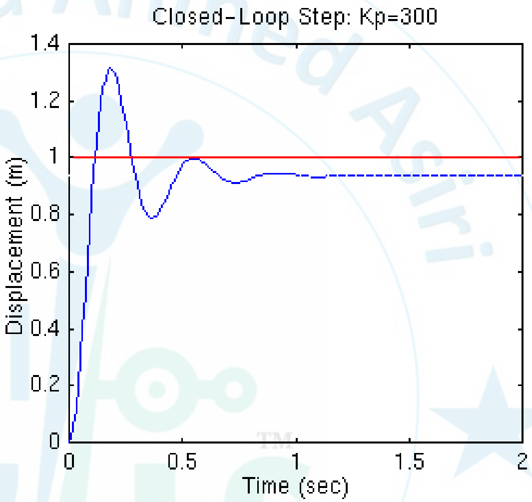
PID Controller



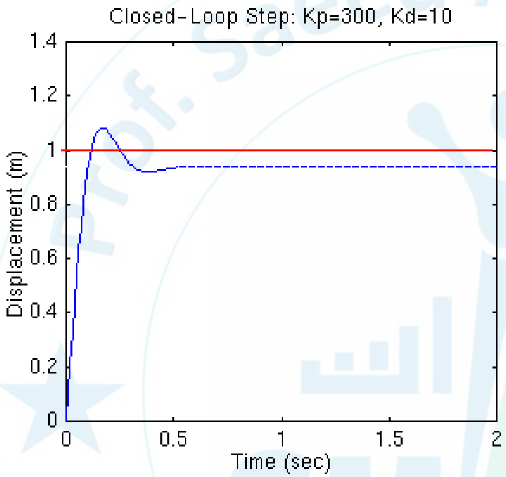
- Let $K_p = 350, \quad K_i = 300, \quad K_d = 5500$
- Now, we have obtained the system with **no overshoot, fast rise time, and no steady-state error.**

Closed Loop System

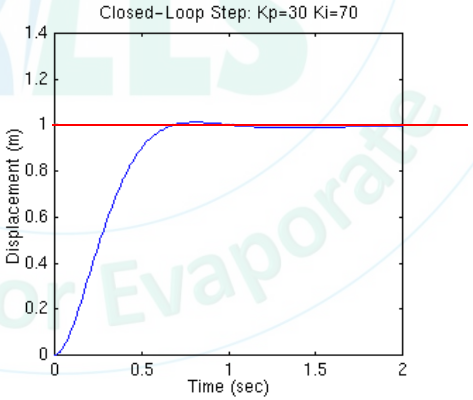
P



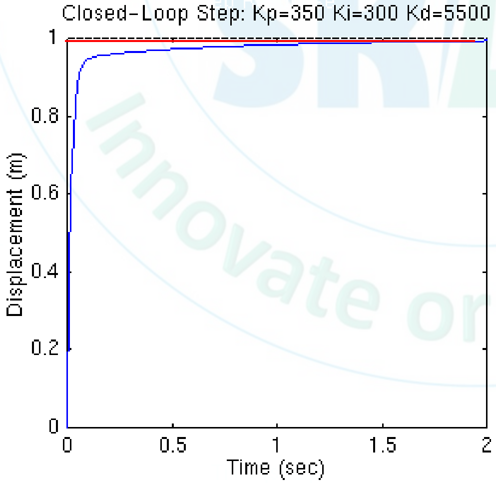
PD



PI



PID



PID Controller Functions

- Output feedback

→ from ***P*roportional action**

compare output with set-point

- Eliminate steady-state offset (=error)

→ from ***I*ntegral action**

apply constant control even when error is zero

- Anticipation

→ From ***D*erivative action**

react to rapid rate of change before errors grows too big

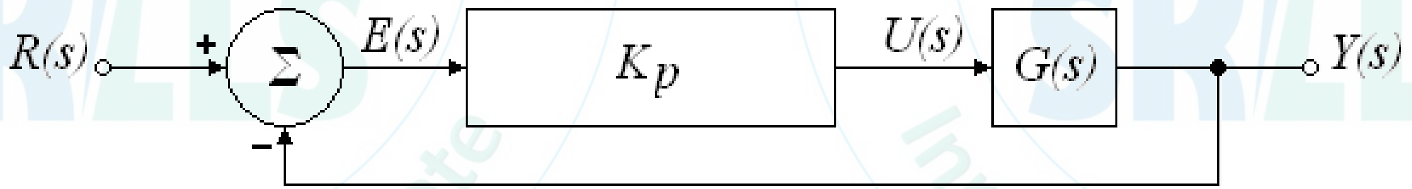
Effect of Proportional, Integral & Derivative Gains on the Dynamic Response

Proportional Controller

- Pure gain (or ***attenuation***) since:

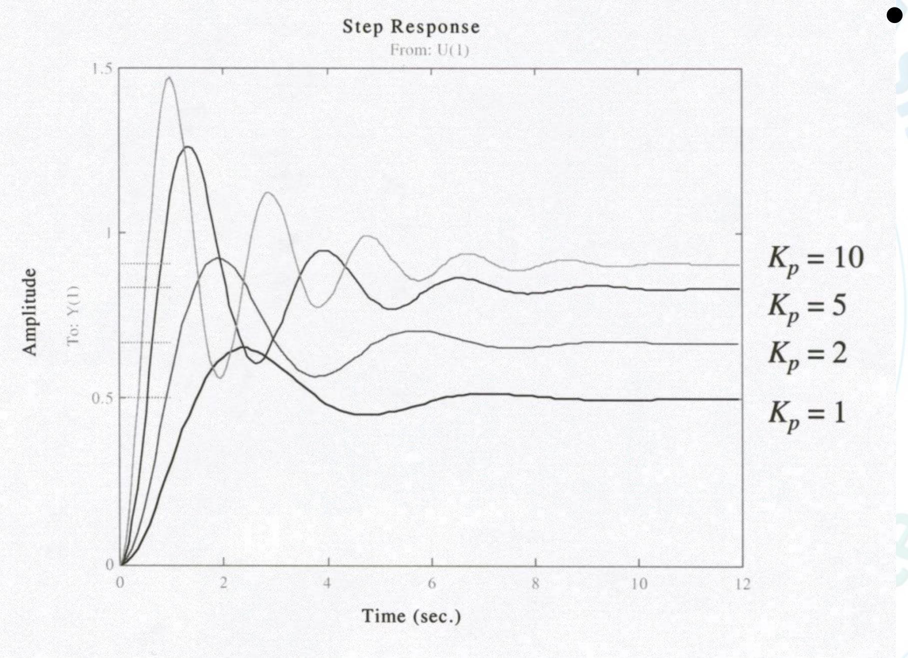
  the controller input is error

  the controller output is a proportional gain



$$E(s)K_p = U(s) \;\Rightarrow\; u(t) = K_p e(t)$$

Change in gain in P controller



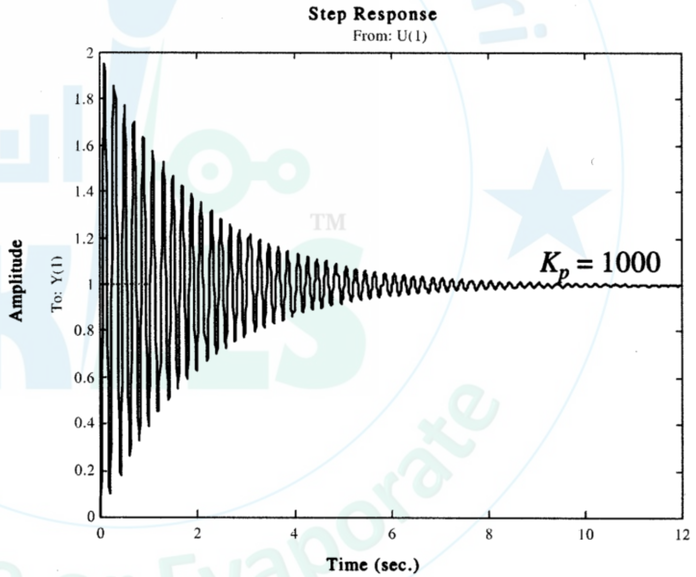
- **Increase in gain:**

  → Upgrade both steady-state and transient responses

  → Reduce steady-state error

  → ***Reduce stability!***

P Controller with *high* gain



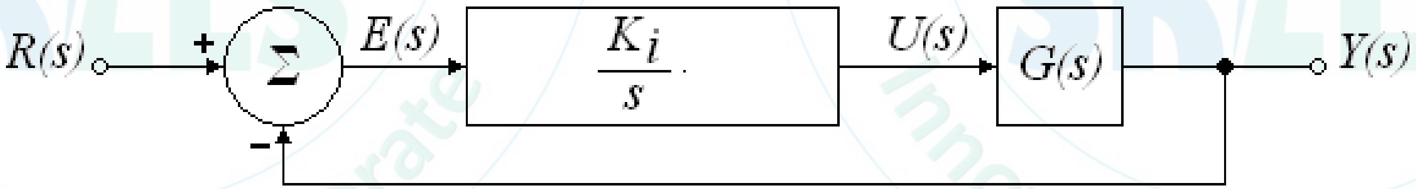
Integral Controller

- Integral of error with a constant gain

$\rightarrow$ increase the system type by 1

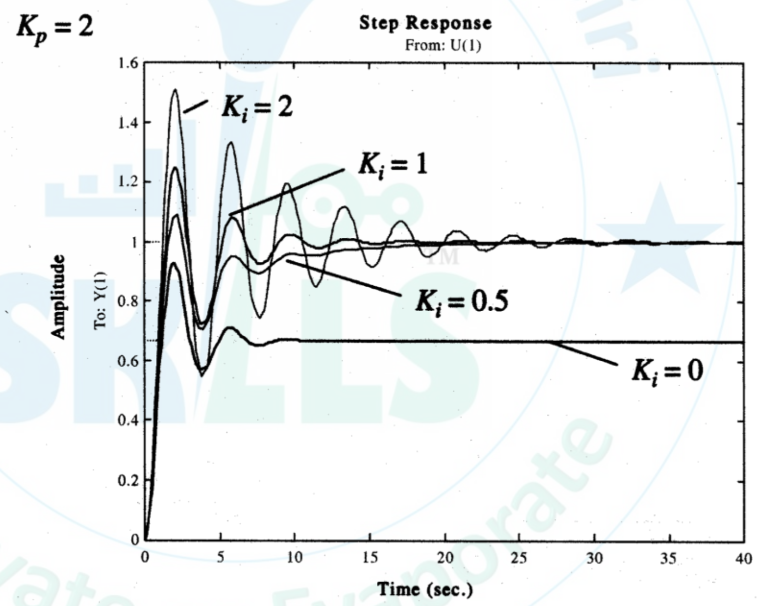
$\rightarrow$ *eliminate steady-state error for a unit step input*

$\rightarrow$ amplify overshoot and oscillations



$$E(s)\frac{K_i}{s} = U(s) \;\Rightarrow\; u(t) = K_i \int_0^t e(t)\,dt$$

Change in gain for PI controller



- **Increase in gain**:
  - → Do not upgrade steady-state responses
  - → Increase slightly settling time
  - → ***Increase oscillations and overshoot!***

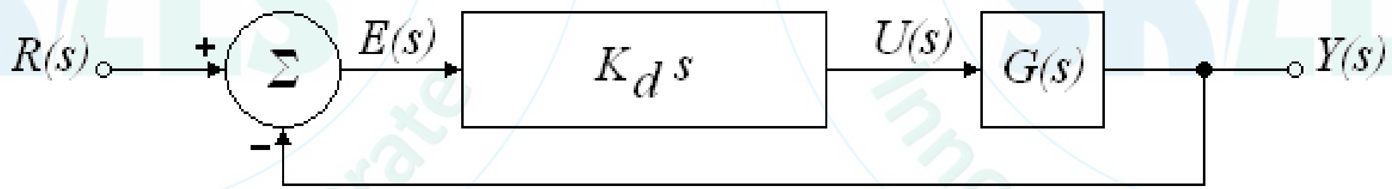
Derivative Controller

- Differentiation of error with a constant gain

$\rightarrow$ detect rapid change in output

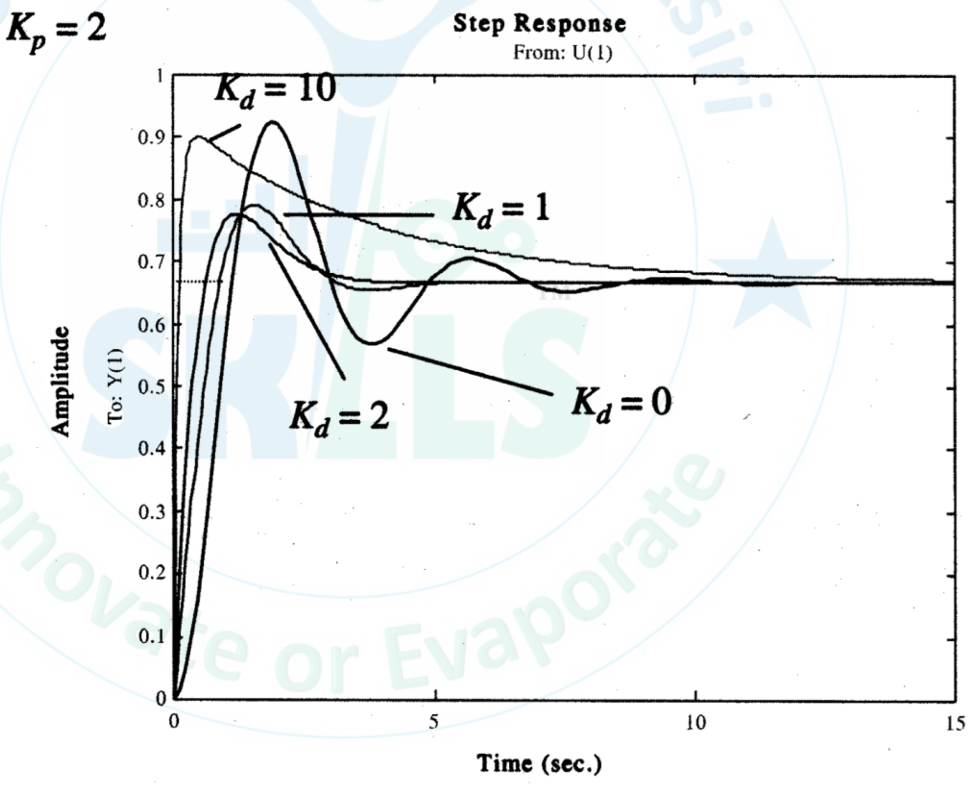
$\rightarrow$ *reduce overshoot and oscillation*

$\rightarrow$ do not affect the steady-state response



$$E(s)K_d s = U(s) \;\Rightarrow\; u(t) = K_d \frac{de(t)}{dt}$$

Effect of change for gain PD controller



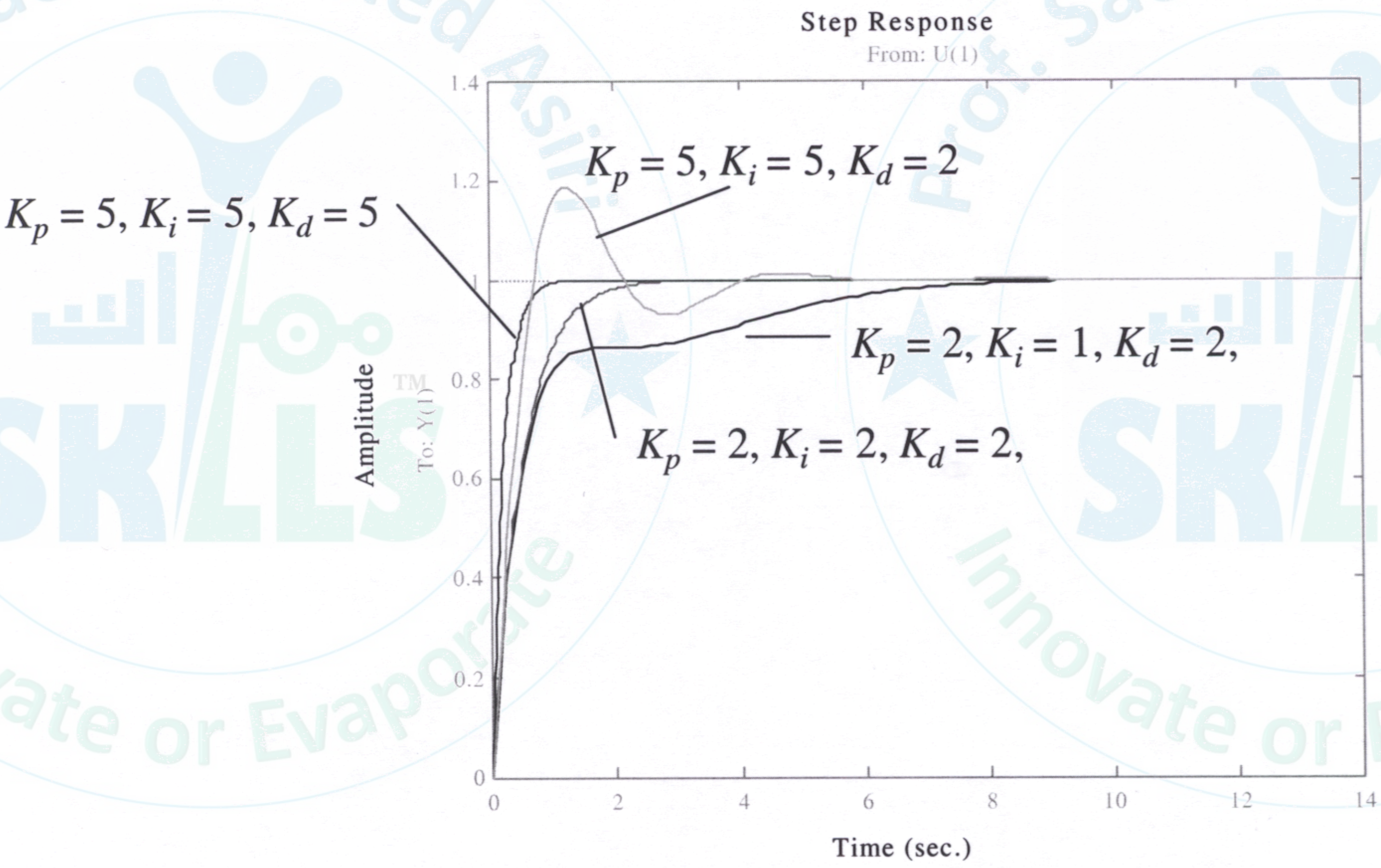
- **Increase in gain**:

  → Upgrade transient response

  → Decrease the peak and rise time

  → ***Increase overshoot and settling time!***

Changes in gains for PID Controller



Conclusions

- **Increasing the proportional feedback gain** ***reduces steady-state errors***, but high gains almost always ***destabilize the system.***
- **Integral control** provides ***robust reduction in steady-state errors***, but often ***makes the system less stable.***
- **Derivative control** usually ***increases damping and improves stability,*** but has almost ***no effect on the steady state error***
- These ***3 kinds of control combined*** from the **classical PID controller**

Conclusion - PID

- The standard PID controller is described by the equation:

$$U(s) = \left(K_p + \frac{K_i}{s} + K_d s\right) E(s)$$

$$or \quad U(s) = K_p\left(1 + \frac{1}{T_i} s + T_d s\right) E(s)$$

Application of PID Control

- PID regulators provide reasonable control of most industrial processes, provided that the performance demands is not too high.

- PI control are generally adequate when plant/process dynamics are essentially of 1^{st} - order.

- PID control are generally ok if dominant plant dynamics are of 2^{nd}-order.

- More elaborate control strategies needed if process has long time delays, or lightly-damped vibrational modes