Root Locus Plots

Goal

Learn a *specific technique* which shows how

changes in one of a system's parameter

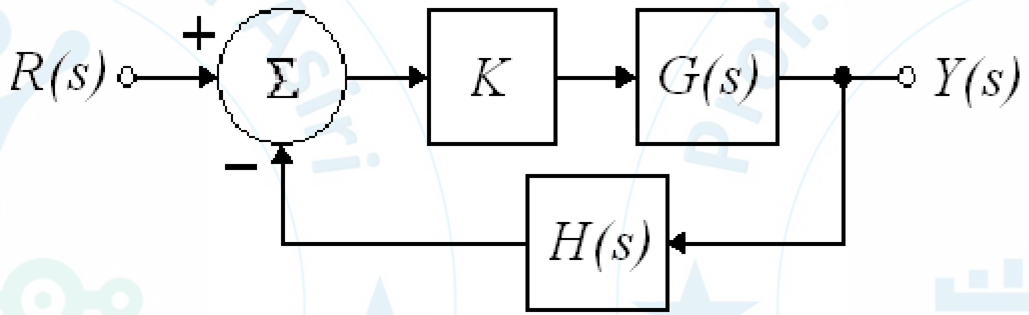
(usually the controller gain, K)

will modify the location of the closed-loop poles

in the s-domain.

Definition

- The closed-loop poles of the negative feedback control:



are the roots of the characteristic equation:

$$\boxed{1 + KG(s)H(s) = 0}$$

The root locus is the locus of the closed-loop poles when a specific parameter (usually gain, K) is varied from 0 to infinity.

Root Locus Method Foundations

- The value of *s* in the *s*-plane that make the loop gain *KG*(s)*H*(s) equal to -1 are the closed-loop poles

  (*i.e.*)

$$1 + KG(s)H(s) = 0 \Leftrightarrow KG(s)H(s) = -1$$

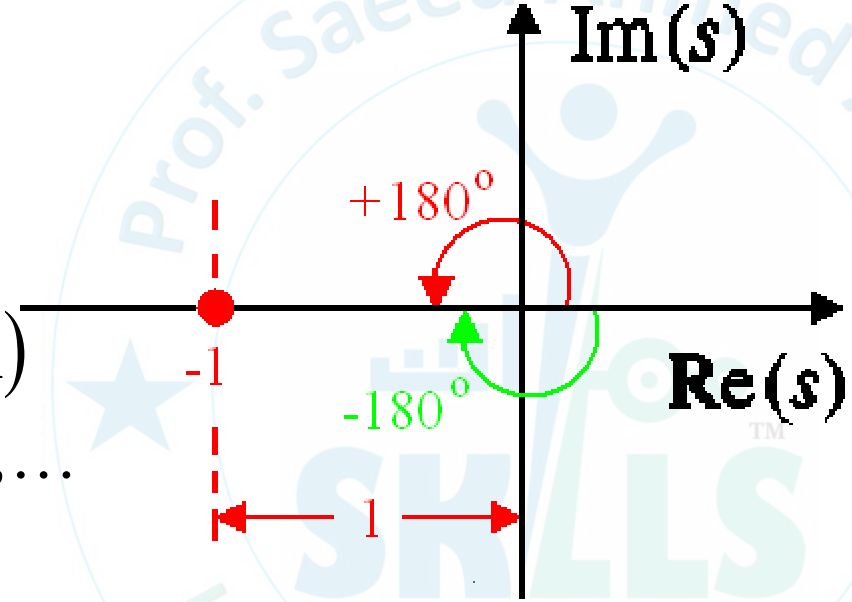
- *KG*(s)*H*(s) = -1 **can be split into two equations by equating the magnitudes and angles** of both sides of the equation.

Angle and Magnitude Conditions

$$KG(s)H(s) = -1$$

$$\Leftrightarrow \begin{cases} |KG(s)H(s)| = 1 \\ \angle KG(s)H(s) = \pm 180^0 (2l+1) \end{cases}$$

$$l = 0, 1, 2, \ldots$$



$$\Leftrightarrow \begin{cases} |G(s)H(s)| = 1/K \\ \boxed{\angle G(s)H(s) = \pm 180^o (2l+1) \qquad l = 0, 1, 2, \ldots} \end{cases}$$

Independent of K

Root Locus

- **Motivation**

  To satisfy transient performance requirements, it may be necessary to know how to choose certain controller parameters so that the resulting closed-loop poles are in the performance regions, which can be solved with Root Locus technique.

- **Definition**

  A graph displaying the roots of a polynomial equation when one of the parameters in the coefficients of the equation changes from 0 to ∞.

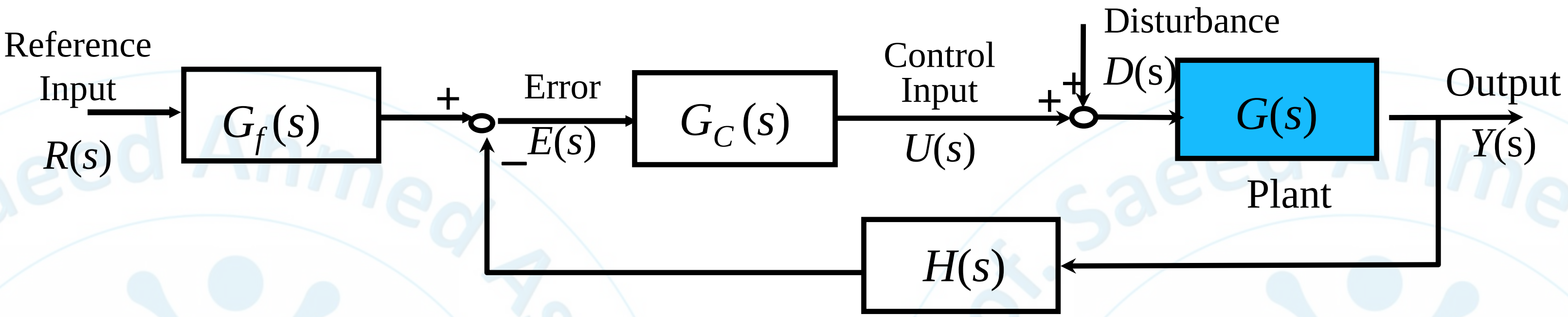
- **Rules for Sketching Root Locus**

- **Examples**

- **Controller Design Using Root Locus**

  Letting the CL characteristic equation (CLCE) be the polynomial equation, one can use the Root Locus technique to find how a positive controller design parameter affects the resulting CL poles, from which one can choose a right value for the controller parameter.

Closed-Loop Characteristic Equation (CLCE)



The closed-loop transfer function $G_{YR}(s)$ is:

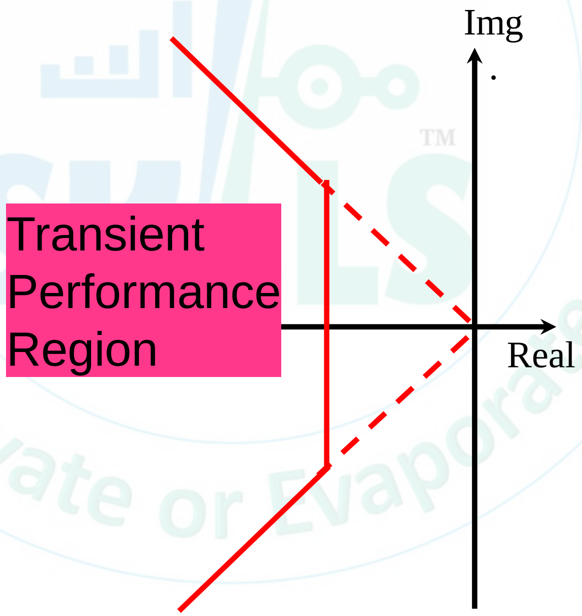
$$G_{YR}(s) = \frac{G(s)G_c(s)G_f(s)}{1+G(s)G_c(s)H(s)}$$

The closed-loop characteristic equation (CLCE) is:

$$1+G(s)G_c(s)H(s)=0$$

For simplicity, assume a simple proportional feedback controller:

$$G_c(s)=K_p \Rightarrow 1+K_pGH=0$$



The transient performance specifications define a region on the complex plane where the closed-loop poles should be located.

Q: How should we choose K_P such that the CL poles are within the desired performance boundary?

Motivation

Ex: From the previous in-class exercise, the closed-loop characteristic equation for the DC motor positioning system under proportional control is:

$$1 + K_P K_S G(s) = 0 \quad \Rightarrow \quad 1 + K_P \cdot 0.03 \cdot \frac{16}{s(0.0174s+1)} = 0$$

Q: ***How to choose K_P such that the resulting closed-loop poles are in the desired performance region?***

- *How do we find the roots of the equation:*

$$1 + K_P \cdot 0.03 \cdot \frac{16}{s(0.0174s+1)} = 0$$

as a function of the design parameter K_P ?

- Graphically display the locations of the closed-loop poles for all $K_P>0$ on the complex plane, from which we know the range of values for K_P that CL poles are in the performance region.

Root Locus – Definition

Root Locus is the method of graphically displaying the roots of a polynomial equation having the following form on the complex plane when the parameter *K* varies from 0 to ∞:

$$1 + K \cdot G(s) = 0 \quad \text{or} \quad 1 + K \cdot \frac{N(s)}{D(s)} = 0$$

where *N*(*s*) and *D*(*s*) are ***known*** polynomials in factorized form:

$$N(s) = (s - z_1)(s - z_2)\cdots(s - z_{N_Z})$$

$$D(s) = (s - p_1)(s - p_2)\cdots(s - p_{N_P})$$

Conventionally, the N_Z roots of the polynomial *N*(*s*) , $z_1, z_2, \ldots, z_{Nz}$, are called the finite *open-loop zeros*. The N_P roots of the polynomial *D*(*s*) , $p_1, p_2, \ldots, p_{Np}$, are called the finite *open-loop poles*.

Note: By transforming the closed-loop characteristic equation of a feedback controlled system with a single positive design parameter *K* into the above standard form, one can use the Root Locus technique to determine the range of *K* that have CL poles in the performance region.

Methods of Obtaining Root Locus

- Given a value of *K*, numerically solve the 1 + *K G*(*s*) = 0 equation to obtain all roots. Repeat this procedure for a set of *K* values that span from 0 to ∞ and plot the corresponding roots on the complex plane. *(This is what we did in the last in-class exercise.)*

- In MATLAB, use the commands *rlocus* and *rlocfind*. A very efficient root locus design tool is the command *rltool*. You can use on-line help to find the usage for these commands.

$$1 + K_P \cdot 0.03 \cdot \frac{16}{s(0.0174s+1)} = 0 \quad \Rightarrow \quad 1 + K_P \cdot \frac{0.48}{0.0174s^2 + s} = 0$$

No open-loop zeros

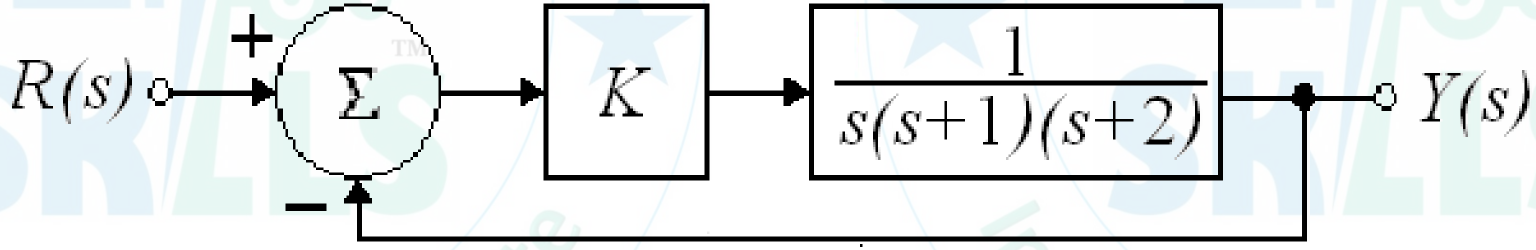
Two open-loop poles

- Apply the following root locus sketching rules to obtain an *approximated* root locus plot.

```
>> op_num=[0.48];
>> op_den=[0.0174 1 0];
>> rlocus(op_num,op_den);
>> [K, poles]=rlocfind(op_num,op_den);
```

Learning by doing – Example 1

1) Sketch the root locus of the following system:



2) Determine the value of K such that the damping ratio ζ of a pair of dominant complex conjugate closed-loop is 0.5.

Rule #1

Assuming *n* poles and *m* zeros for *G*(*s*)*H*(*s*):

- The ***n* branches** of the root locus **start at the *n* poles.**
- ***m*** of these *n* branches **end on the *m* zeros**
- The ***n-m* other branches terminate at infinity** along asymptotes.

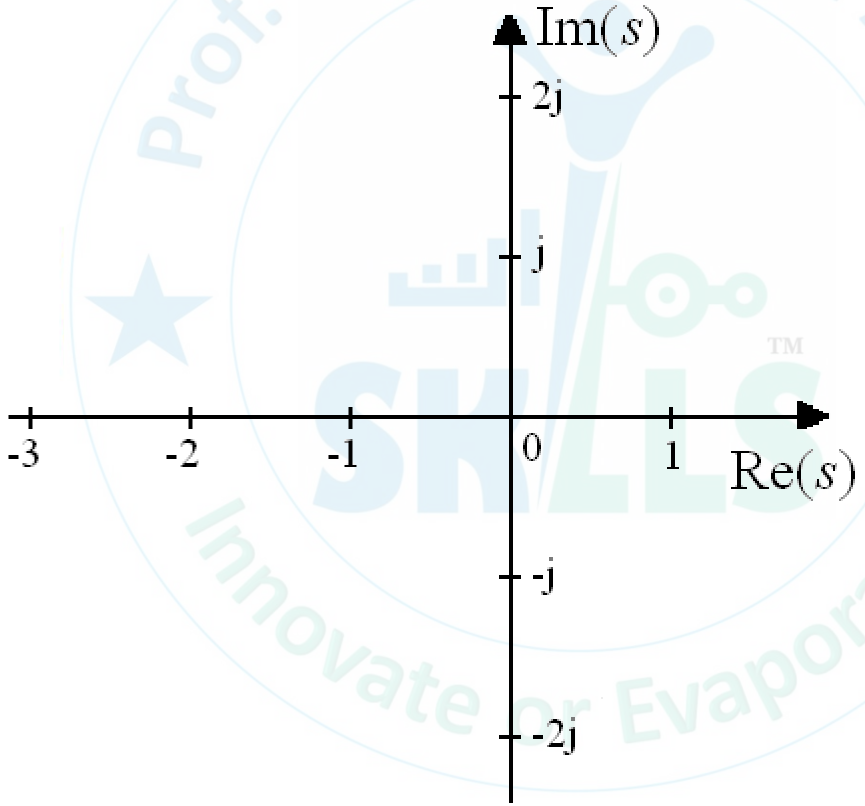
First step: Draw the *n* poles and *m* zeros of *G*(*s*)*H*(*s*) using x and o respectively

Applying Step #1

Draw the n poles and m zeros of $G(s)H(s)$ using x and o respectively.

$$G(s)H(s) = \frac{1}{s(s+1)(s+2)}$$

- 3 poles:
  $p_1 = 0$; $p_2 = -1$; $p_3 = -2$
- No zeros



Applying Step #1

Draw the *n* poles and *m* zeros of *G*(*s*)*H*(*s*) using x and o respectively.

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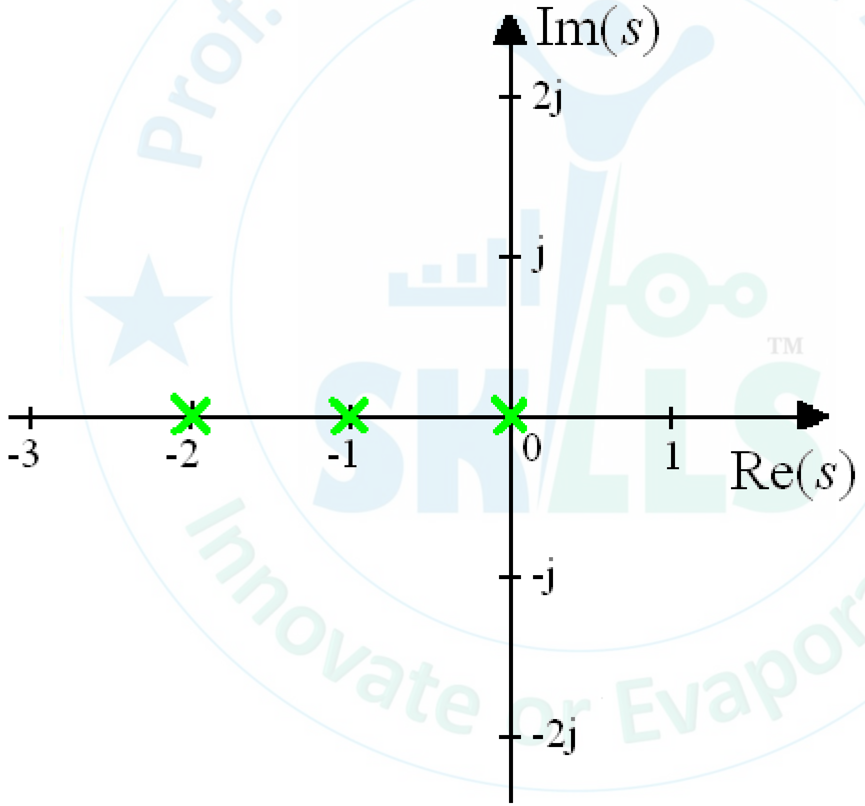
- No zeros



Rule #2

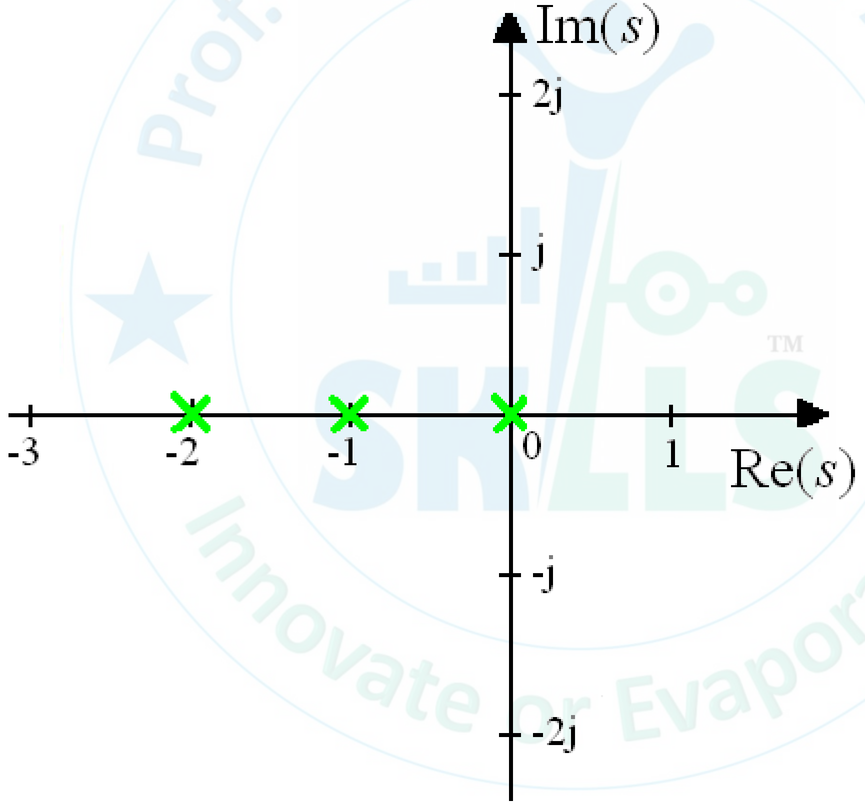
The **loci on the real axis** are to the **left of an ODD number of REAL poles and REAL zeros** of $G(s)H(s)$

Second step: Determine the loci on the real axis. Choose a arbitrary test point. If the TOTAL number of both real poles and zeros is to the RIGHT of this point is ODD, then this point is on the root locus

Applying Step #2

Determine the loci on the real axis:

- Choose a arbitrary test point.
- If the TOTAL number of both real poles and zeros is to the RIGHT of this point is ODD, then this point is on the root locus



Applying Step #2

Determine the loci on the real axis:

- Choose a arbitrary test point.
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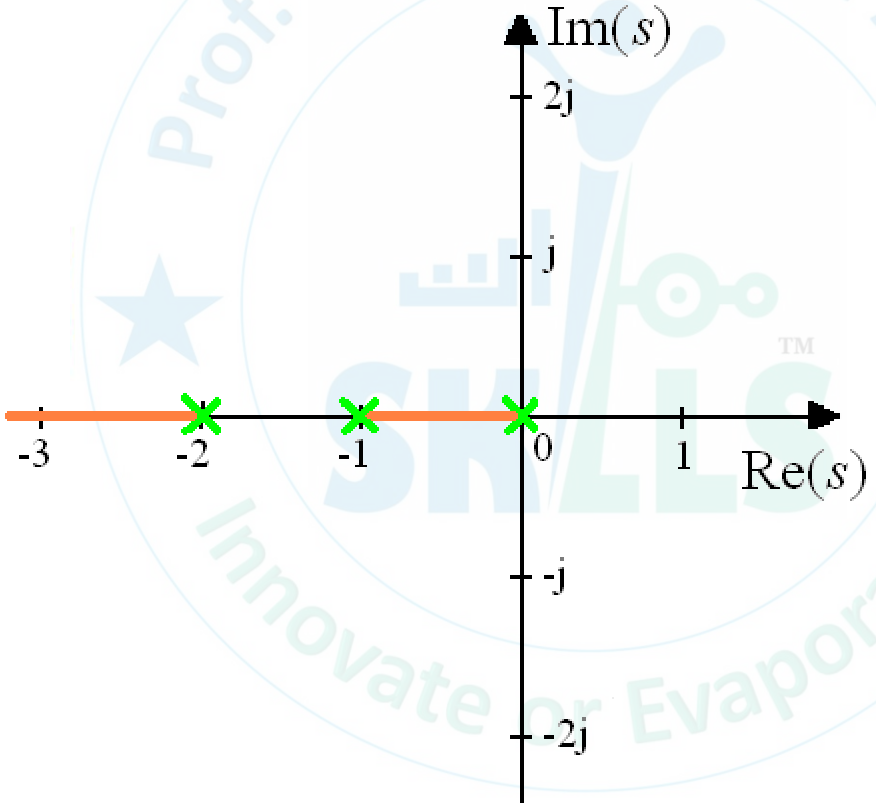


Rule #3

Assuming *n* poles and *m* zeros for *G*(*s*)*H*(*s*):

- The **root loci for very large values of s** must be **asymptotic** to **straight lines originate on the real axis at point:**

$$s = \alpha = \frac{\sum_{n} p_i - \sum_{m} z_i}{n - m}$$

radiating out from this point at angles:

$$\phi_l = \frac{\pm 180^o (2l + 1)}{n - m}$$

Third step: Determine the *n* - *m* asymptotes of the root loci. Locate s = α on the real axis. Compute and draw angles. Draw the asymptotes using dash lines.

Applying Step #3

Determine the *n* - *m* asymptotes:

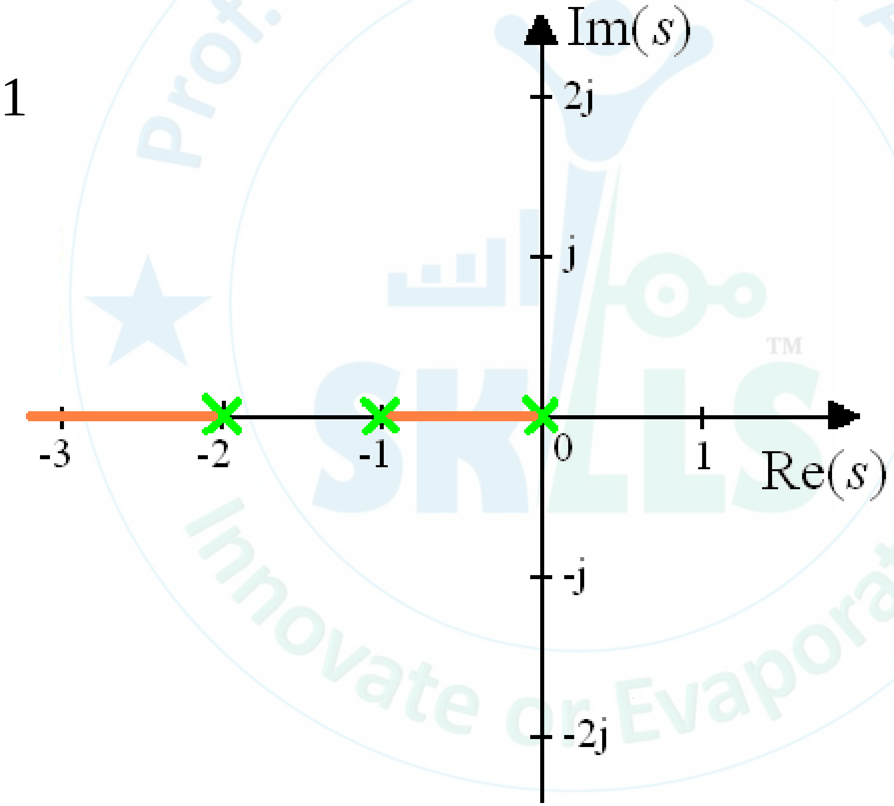
- Locate s = α on the real axis:

$$s = \alpha = \frac{p_1 + p_2 + p_3}{3-0} = \frac{0-1-2}{3} = -1$$

- Compute and draw angles:

$$\phi_l = \frac{\pm 180(2l+1)}{n-m} \quad l = 0,1,2,\dots$$

$$\Rightarrow \begin{cases} \phi_0 = \dfrac{\pm 180^0(2\times 0+1)}{3-0} = \pm 60^0 \\ \phi_1 = \dfrac{\pm 180^0(2\times 1+1)}{3-0} = \pm 180^0 \end{cases}$$



- Draw the asymptotes using dash lines.

Applying Step #3

Determine the *n - m* asymptotes:

- Locate s = α on the real axis:

$$s = \alpha = \frac{p_1 + p_2 + p_3}{3-0} = \frac{0-1-2}{3} = -1$$

- Compute and draw angles:

$$\phi_l = \frac{\pm 180(2l+1)}{n-m} \quad l = 0,1,2,\dots$$

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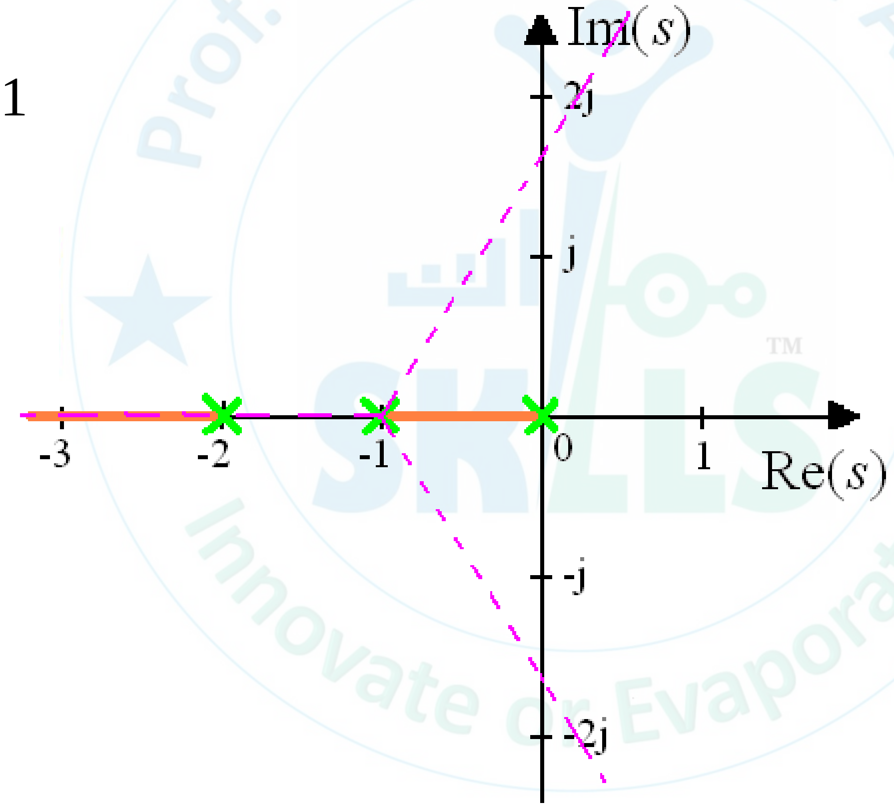
- Draw the asymptotes using dash lines.



Breakpoint Definition

- The breakpoints are the points in the s-domain where **multiples** roots of the characteristic equation of the feedback control occur.

- These points correspond to intersection points on the root locus.

Rule #4

Given the characteristic equation is $KG(s)H(s) = -1$

- The **breakpoints are the closed-loop poles that satisfy:**

$$\frac{dK}{ds} = 0$$

Fourth step: Find the breakpoints. Express K such as:

$$K = \frac{-1}{G(s)H(s)}.$$

Set $dK/ds = 0$ and solve for the poles.

Applying Step #4

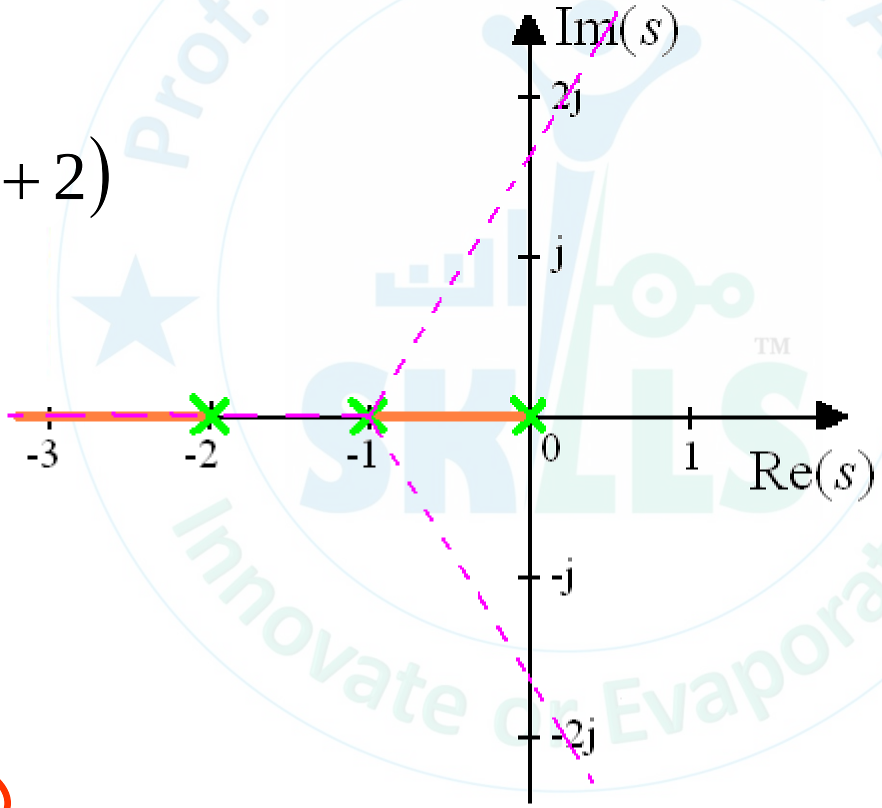
Find the breakpoints.

- Express K such as:

$$K = \frac{-1}{G(s)H(s)} = -s(s+1)(s+2)$$

$$K = -s^3 - 3s^2 - 2s$$

- Set $dK/ds = 0$ and solve for the poles.



$$-3s^2 - 6s - 2 = 0$$

$$\cancel{s_1 = -1.5774}, \quad s_2 = -0.4226$$

Applying Step #4

Find the breakpoints.

- Express K such as:

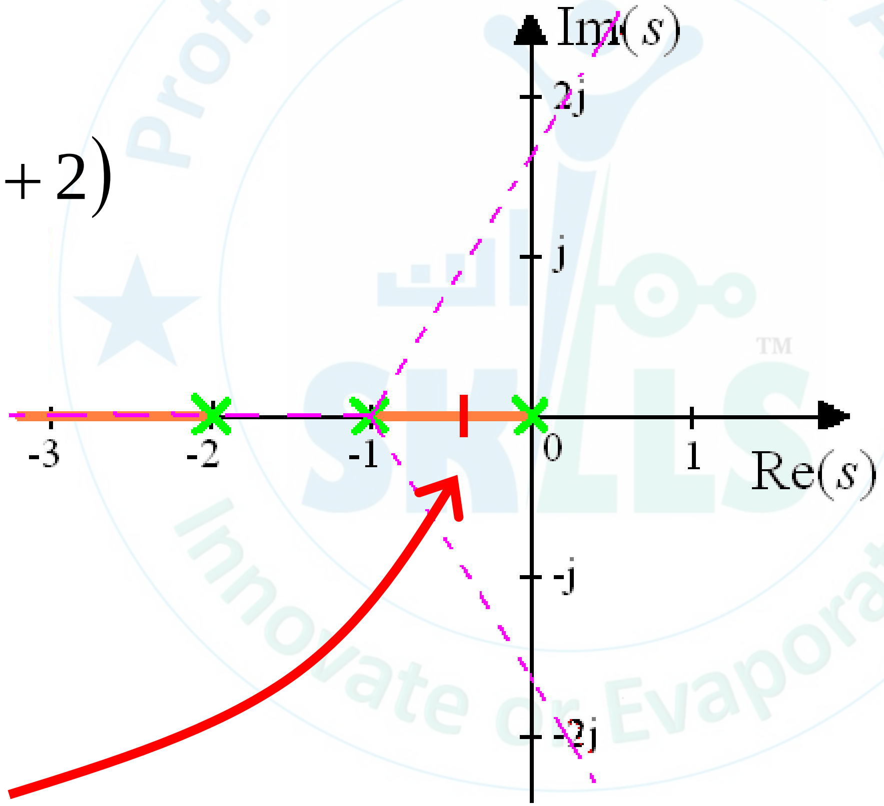
$$K = \frac{-1}{G(s)H(s)} = -s(s+1)(s+2)$$

$$K = -s^3 - 3s^2 - 2s$$

- Set $dK/ds = 0$ and solve for the poles.

$$-3s^2 - 6s - 2 = 0$$

$$\cancel{s_1 = -1.5774}, \quad s_2 = -0.4226$$



Recall Rule #1

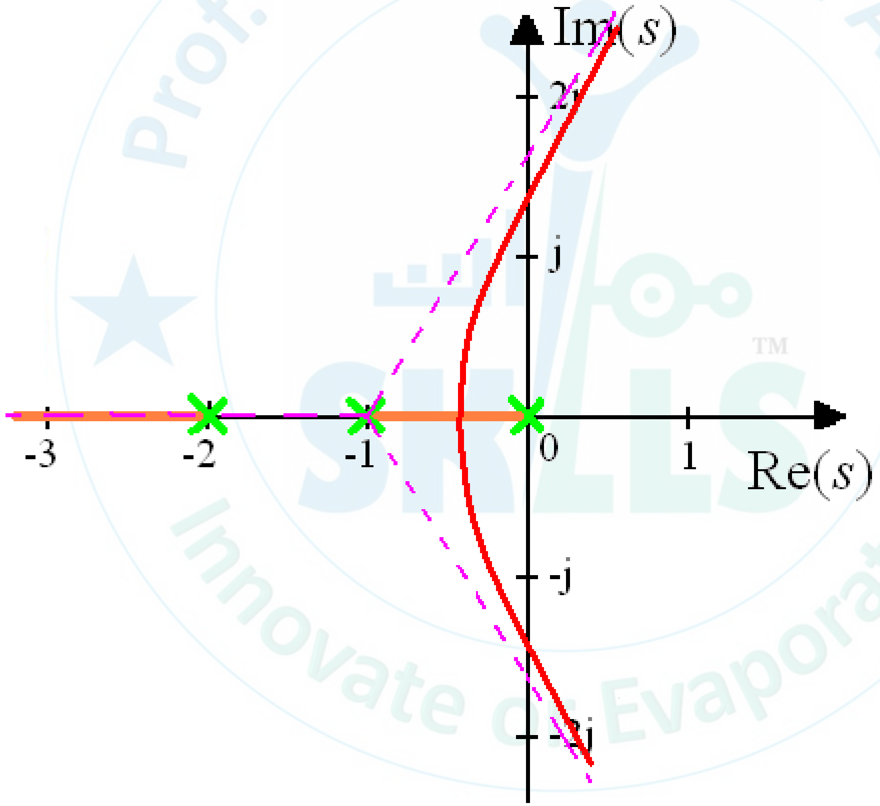
Assuming *n* poles and *m* zeros for *G*(*s*)*H*(*s*):

- The ***n* branches** of the root locus **start at the *n* poles.**
- ***m*** of these *n* branches **end on the *m* zeros**
- The ***n-m* other branches terminate at infinity** along asymptotes.

Last step: Draw the *n-m* branches that terminate at infinity along asymptotes

Applying Last Step

Draw the *n-m* branches that terminate at infinity along asymptotes



Points on both root locus & imaginary axis?

- Points on imaginary axis satisfy:

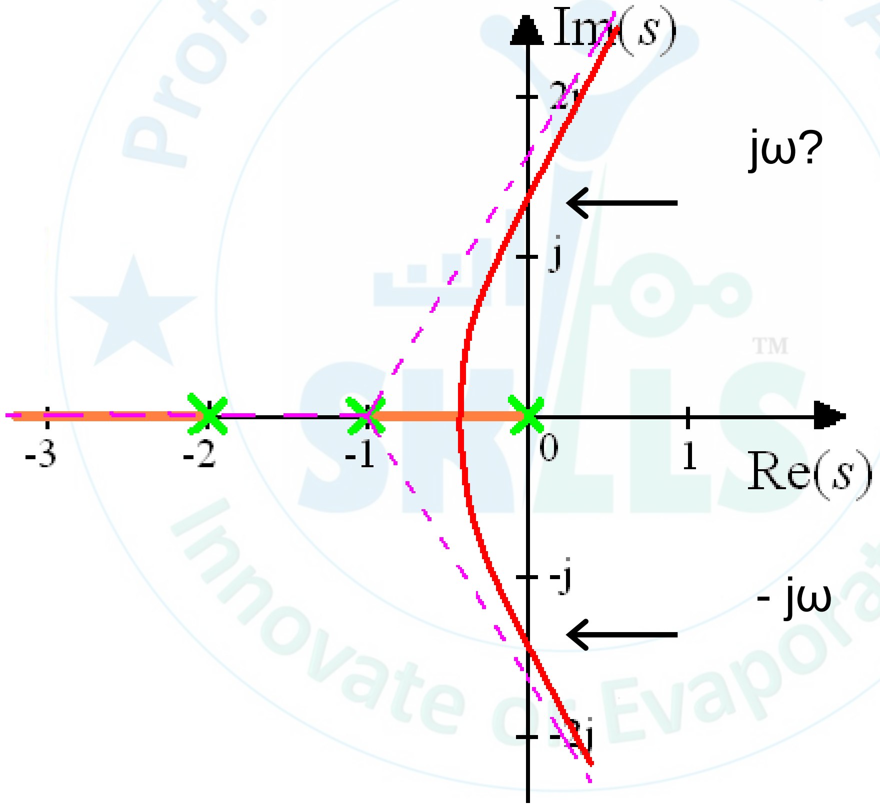
$$s = j\omega$$

- Points on root locus satisfy:

$$1 + KG(s)H(s) = 0$$

- Substitute *s=jω* into the characteristic equation and solve for *ω*.

$$\omega = 0 \quad or \quad \omega = \pm\sqrt{2}$$



Root Locus Sketching Rules

$$1+K\cdot\frac{N(s)}{D(s)}=0 \quad\Rightarrow\quad 1+K\cdot\frac{(s-z_1)(s-z_2)\cdots(s-z_{N_Z})}{(s-p_1)(s-p_2)\cdots(s-p_{N_P})}=0$$

Rule 1: The number of branches of the root locus is equal to the number of closed-loop poles (or roots of the characteristic equation). In other words, the number of branches is equal to the number of open-loop poles or open-loop zeros, whichever is greater. $D(s)+KN(s)=0$

Rule 2: Root locus starts at open-loop poles (*when K*= 0) and ends at open-loop zeros (*when K*=∞). If the number of open-loop poles is greater than the number of open-loop zeros, some branches starting from finite open-loop poles will terminate at zeros at infinity (i.e., go to infinity). If the reverse is true, some branches will start at poles at infinity and terminate at the finite open-loop zeros.

Rule 3: Root locus is symmetric about the real axis, which reflects the fact that closed-loop poles appear in complex conjugate pairs.

Rule 4: Along the real axis, the root locus includes all segments that are to the left of an *odd* number of finite real open-loop poles and zeros.

Check the phases

$$\angle K\frac{N(s)}{D(s)}=\angle -1=\pi[\text{rad}]=180^\circ$$

Root Locus Sketching Rules

Rule 5: If number of poles N_P exceeds the number of zeros N_Z , then as $K \to \infty$, (N_P - N_Z) branches will become asymptotic to straight lines. These straight lines intersect the real axis with angles θ_k at σ_0 .

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = \frac{\text{Sum of open-loop poles } - \text{ Sum of open-loop zeros}}{\text{\# of open-loop poles } - \text{ \# of open-loop zeros}}$$

$$\theta_k = (2k+1)\frac{\pi}{N_P - N_Z}\ [\text{rad}] = (2k+1)\frac{180°}{N_P - N_Z}\ [\text{deg}],\ \ k = 0,\ 1,\ 2,\ \cdots$$

If N_Z exceeds N_P, then as $K \to 0$, (N_Z - N_P) branches behave as above.

Rule 6: Breakaway and/or break-in (arrival) points should be the solutions to the following equations:

$$\frac{d}{ds}\left(\frac{N(s)}{D(s)}\right) = 0 \ \text{ or } \ \frac{d}{ds}\left(\frac{D(s)}{N(s)}\right) = 0$$

Root Locus Sketching Rules

Rule 7: The departure angle for a pole p_i (the arrival angle for a zero z_i) can be calculated by slightly modifying the following equation:

angle criterion ⟶ $$\angle(s-z_1)+\angle(s-z_2)+\cdots+\angle(s-z_{N_z})-\angle(s-p_1)-\angle(s-p_2)-\cdots-\angle(s-p_{N_p})=180^\circ$$

The departure angle q_j from the pole p_j can be calculated by replacing the term $\angle(s-p_j)$ with q_j and replacing all the s's with p_j in the other terms.

Rule 8: If the root locus passes through the imaginary axis (the stability boundary), the crossing point $j\omega$ and the corresponding gain K can be found as follows:

- Replace s in the left side of the closed-loop characteristic equation with $j\omega$ to obtain the real and imaginary parts of the resulting complex number.
- Set the real and imaginary parts to zero, and solve for ω and K. This will tell you at what values of K and at what points on the $j\omega$ axis the roots will cross.

magnitude criterion ⟶ $$K=\frac{|s-p_1||s-p_2|\cdots|s-p_{N_P}|}{|s-z_1||s-z_2|\cdots|s-z_{N_z}|}$$

Steps to Sketch Root Locus

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1+K\cdot\frac{N(s)}{D(s)}=0 \quad \text{or} \quad 1+K\cdot\frac{(s-z_1)(s-z_2)\cdots(s-z_{N_Z})}{(s-p_1)(s-p_2)\cdots(s-p_{N_P})}=0$$

Step 2: Find the open-loop zeros, z_i, and the open-loop poles, p_i. Mark the open-loop poles and zeros on the complex plane. Use × to represent open-loop poles and ○ to represent the open-loop zeros.

Step 3: Determine the real axis segments that are on the root locus by applying Rule 4.

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

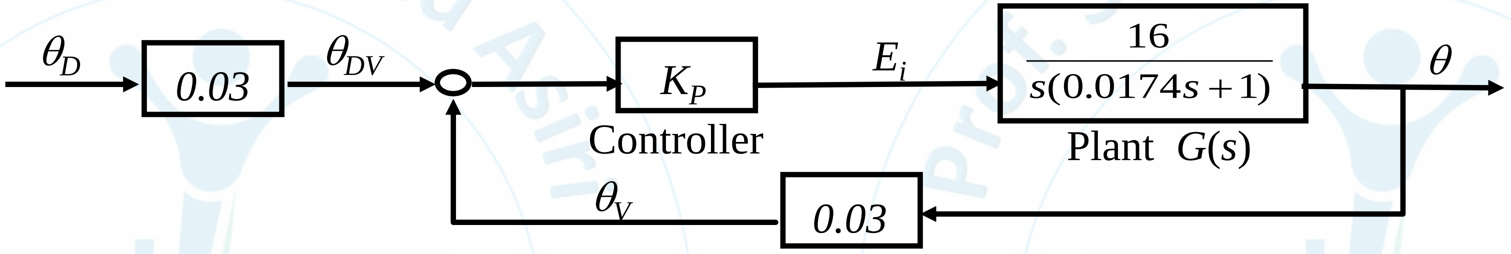
Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.

Example 1

DC Motor Position Control

In the previous example on the printer paper advance position control, the proportional control block diagram is:



Sketch the root locus of the closed-loop poles as the proportional gain K_P varies from 0 to ∞.

Find closed-loop characteristic equation:

$$1+K_pG(s)H(s)=0$$

$$1+K_p\frac{\underset{N(s)}{0.48}}{\underbrace{s(0.0174s+1)}_{D(s)}}=0$$

Example 1

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1+\underbrace{\boxed{27.58K_p}}_{K}\,\frac{\overbrace{1}^{N(s)}}{\underbrace{s(s+57.47)}_{D(s)}}=0$$

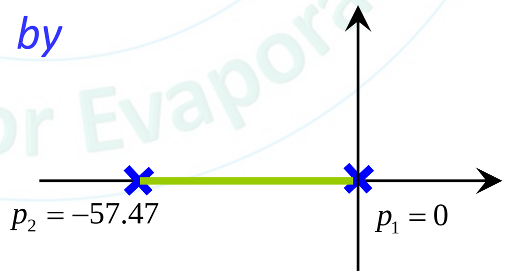
Step 2: Find the open-loop zeros, z_i, and the open-loop poles, p_i :

No open-loop zeros

open-loop poles

$$p_1=0,\ p_2=-57.47$$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



Example 1

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = \frac{-57.47}{2} = -28.74$$

$$\theta_k = (2k+1)\frac{\pi}{N_P - N_Z} \ \text{[rad]} \qquad = \begin{cases} \frac{\pi}{2} \\ \frac{3\pi}{2} \end{cases}$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds}\left(\frac{N(s)}{D(s)}\right) = 0 \ \text{ or } \ \frac{d}{ds}\left(\frac{D(s)}{N(s)}\right) = 0,$$

$$\frac{d}{ds}\left(\frac{s(0.0174s+1)}{0.48}\right) = 0, 0.0348s + 1 = 0, s = -28.74$$

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

$$-\angle(p_2 - p_1) - \theta_{p_2} = 180^\circ, \theta_{p_2} = 0^\circ$$

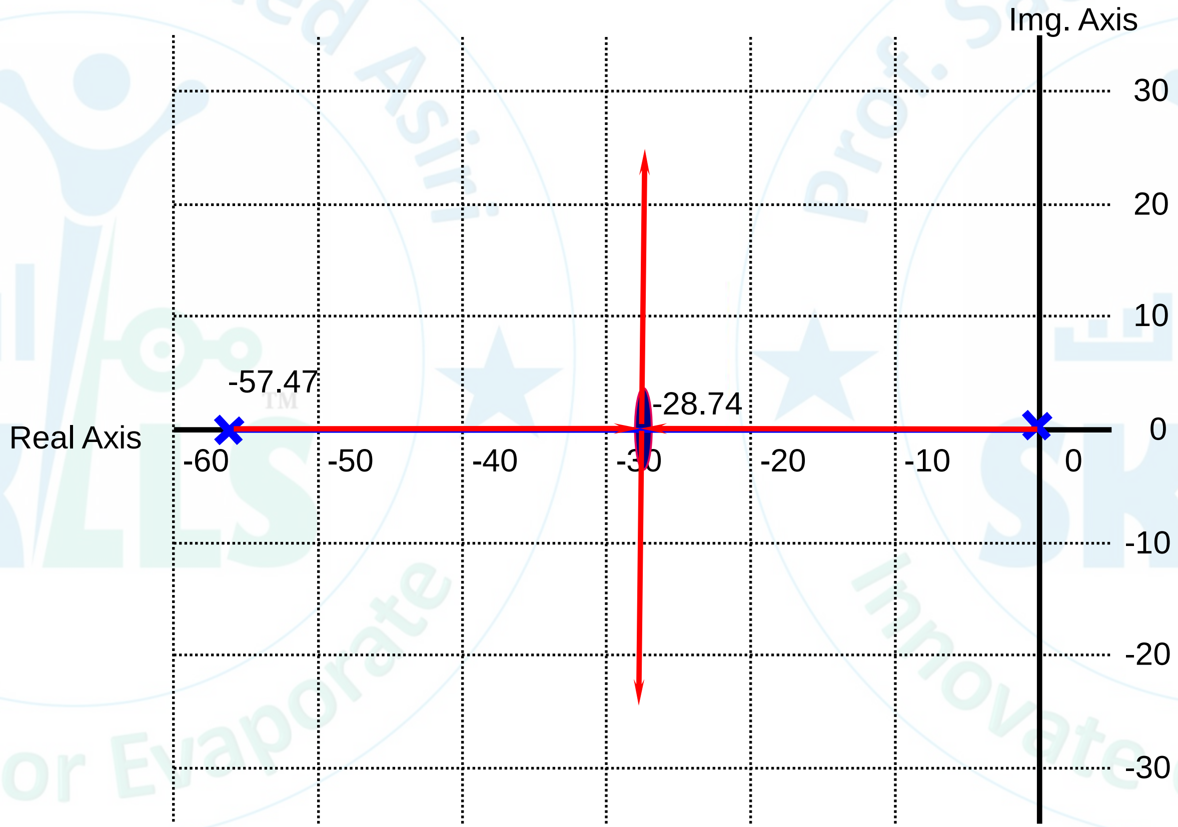
$$-\theta_{p_1} - \angle(p_1 - p_2) = 180^\circ, \theta_{p_1} = 180^\circ$$

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

Could s be pure imaginary in this example?

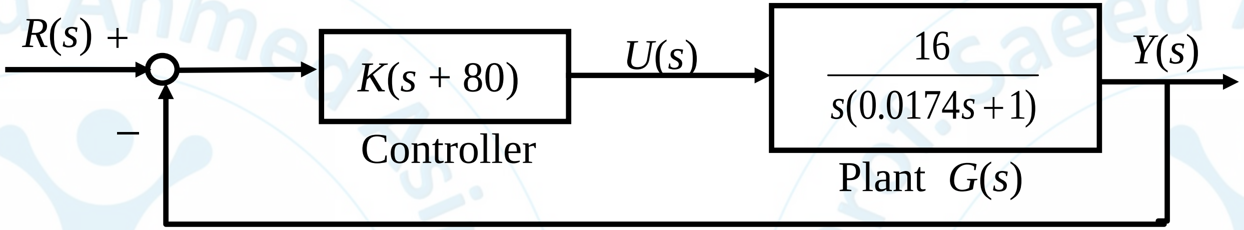
Example 1

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.



Example 2

A positioning feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain K varies from 0 to ∞.

Find closed-loop characteristic equation:

$$1+G_c(s)G(s)H(s)=0$$

$$1+K(s+80)\frac{16}{s(0.0174s+1)}=0$$

Example 2

Step 1: Formulate the (closed-loop) characteristic equation into the standard form for sketching root locus:

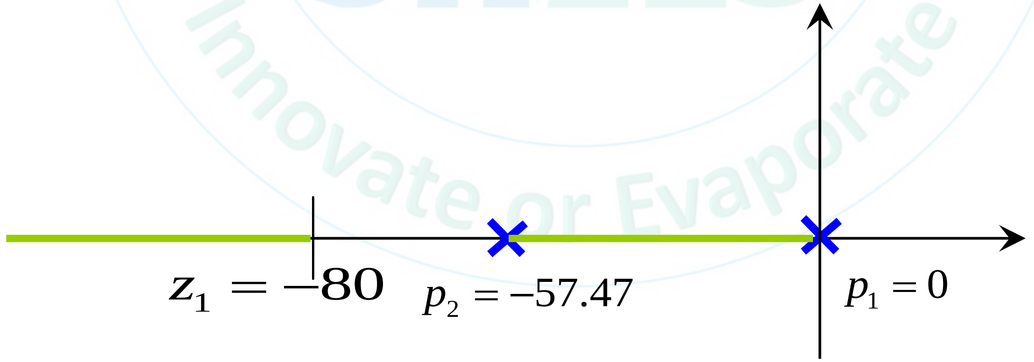
$$1+K\frac{\overbrace{16(s+80)}^{N(s)}}{\underbrace{s(0.0174s+1)}_{D(s)}}=1+\overbrace{\boxed{920K}}^{K}\frac{\overbrace{(s+80)}^{N(s)}}{\underbrace{s(s+57.47)}_{D(s)}}=0$$

Step 2: Find the open-loop zeros, z_i, and the open-loop poles, p_i :

open-loop zeros $\quad z_1=-80$

open-loop poles $\quad p_1=0,\ p_2=-57.47$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



Example 2

Step 4: *Determine the number of asymptotes and the corresponding intersection* σ_0 *and angles* θ_k *by applying Rules 2 and 5.*

Step 5: *(If necessary) Determine the break-away and break-in points using Rule 6.*

$$\frac{d}{ds}\left(\frac{N(s)}{D(s)}\right)=0 \quad \text{or} \quad \frac{d}{ds}\left(\frac{D(s)}{N(s)}\right)=0,$$

$$\frac{d}{ds}\left(\frac{(s+80)}{s(s+57.47)}\right)= \qquad \frac{s(s+57.47)-(s+80)(2s+57.47)}{\left[s(s+57.47)\right]^2}=0,$$

$$s^2+160s+4600=0$$

$$s_1=-122, s_2=-37.6$$

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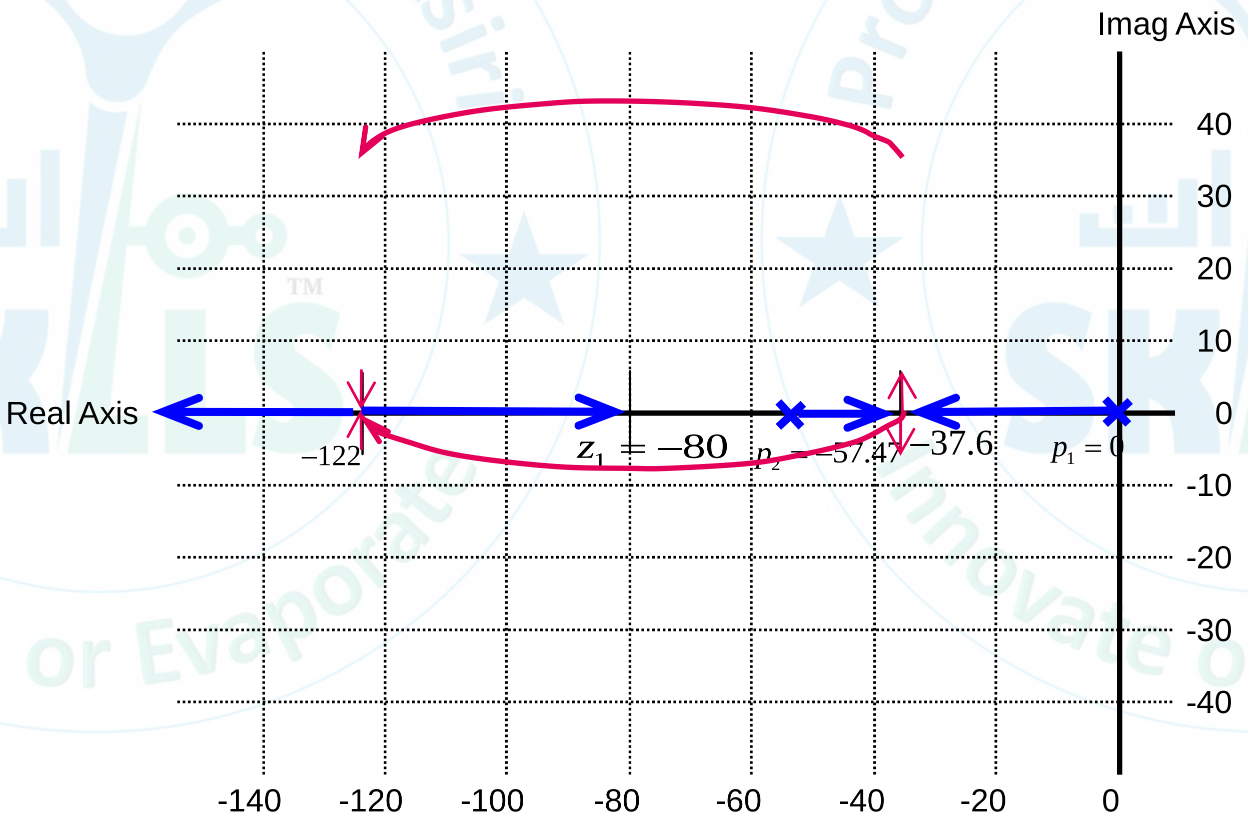
Example 2

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

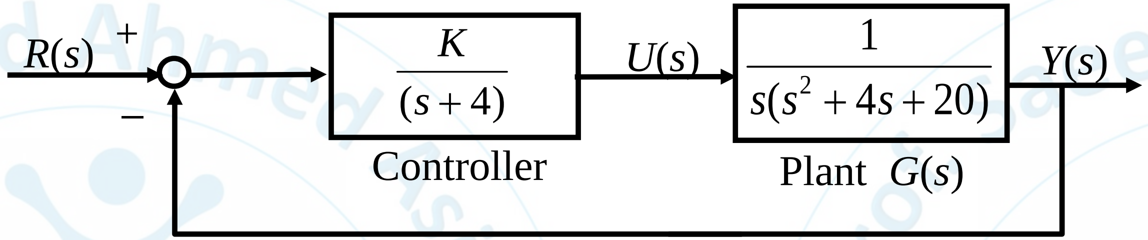
Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.



Example 3

A feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain *K* varies from 0 to ∞.

Find closed-loop characteristic equation:

$$1+G_c(s)G(s)H(s)=0$$

$$1+\frac{K}{s+4}\,\frac{1}{s\left(s^2+4s+20\right)}=0$$

Example 3

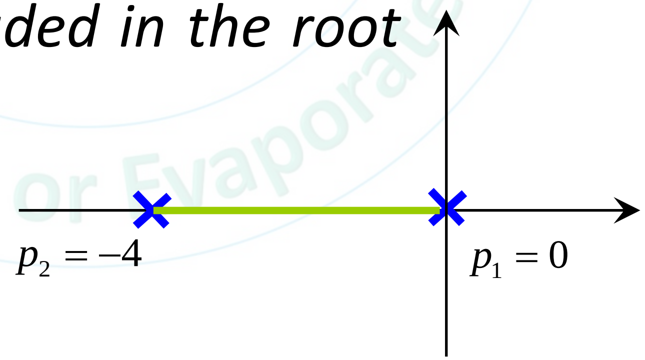
Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1+K\overbrace{\frac{1}{\underbrace{s\left(s^2+4s+20\right)\left(s+4\right)}_{D(s)}}}^{N(s)}=0$$

Step 2: Find the open-loop zeros, z_i, and the open-loop poles, p_i:

open-loop zeros	No open-loop zeros
open-loop poles	$p_1=0,\ p_2=-4,\ p_{3,4}=-2\pm 4j$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



Example 3

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = \frac{0 + (-4) + (-2+4j) + (-2-4j)}{4-0} = -2$$

$$\theta_k = (2k+1)\frac{\pi}{N_P - N_Z}\ [\text{rad}] = \begin{cases} \frac{\pi}{4} \\ \frac{3\pi}{4} \\ \frac{5\pi}{4} \\ \frac{7\pi}{4} \end{cases}$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds}\left(\frac{N(s)}{D(s)}\right) = 0 \ \text{or} \ \frac{d}{ds}\left(\frac{D(s)}{N(s)}\right) = 0,$$

$$\frac{d}{ds}\left(\frac{D(s)}{N(s)}\right) = \frac{d}{ds}\left(\frac{s(s^2+4s+20)(s+4)}{1}\right) = \frac{d}{ds}(s^4 + 8s^3 + 36s^2 + 80s)$$

$$= 4s^3 + 24s^2 + 72s + 80 = 0$$

$$s_1 = -2, s_{2,3} = -2 \pm 2.45j$$

Example 3

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

$$\sum_{i=1}^{N_z}\angle(s-z_i)-\sum_{i=1}^{N_p}\angle(s-p_i)=180^\circ$$

$$p_1=0: \quad \theta_{p_1}=180^\circ \qquad p_3=-2+4j: \quad \theta_{p_3}=-90^\circ$$

$$p_2=-4: \quad \theta_{p_2}=0^\circ \qquad p_4=-2-4j: \quad \theta_{p_4}=90^\circ$$

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

$$1+K\frac{1}{s\left(s^2+4s+20\right)\left(s+4\right)}=0 \Rightarrow \quad s\left(s^2+4s+20\right)\left(s+4\right)+K=0$$

$$\Leftrightarrow s^4+8s^3+36s^2+80s+K=0$$

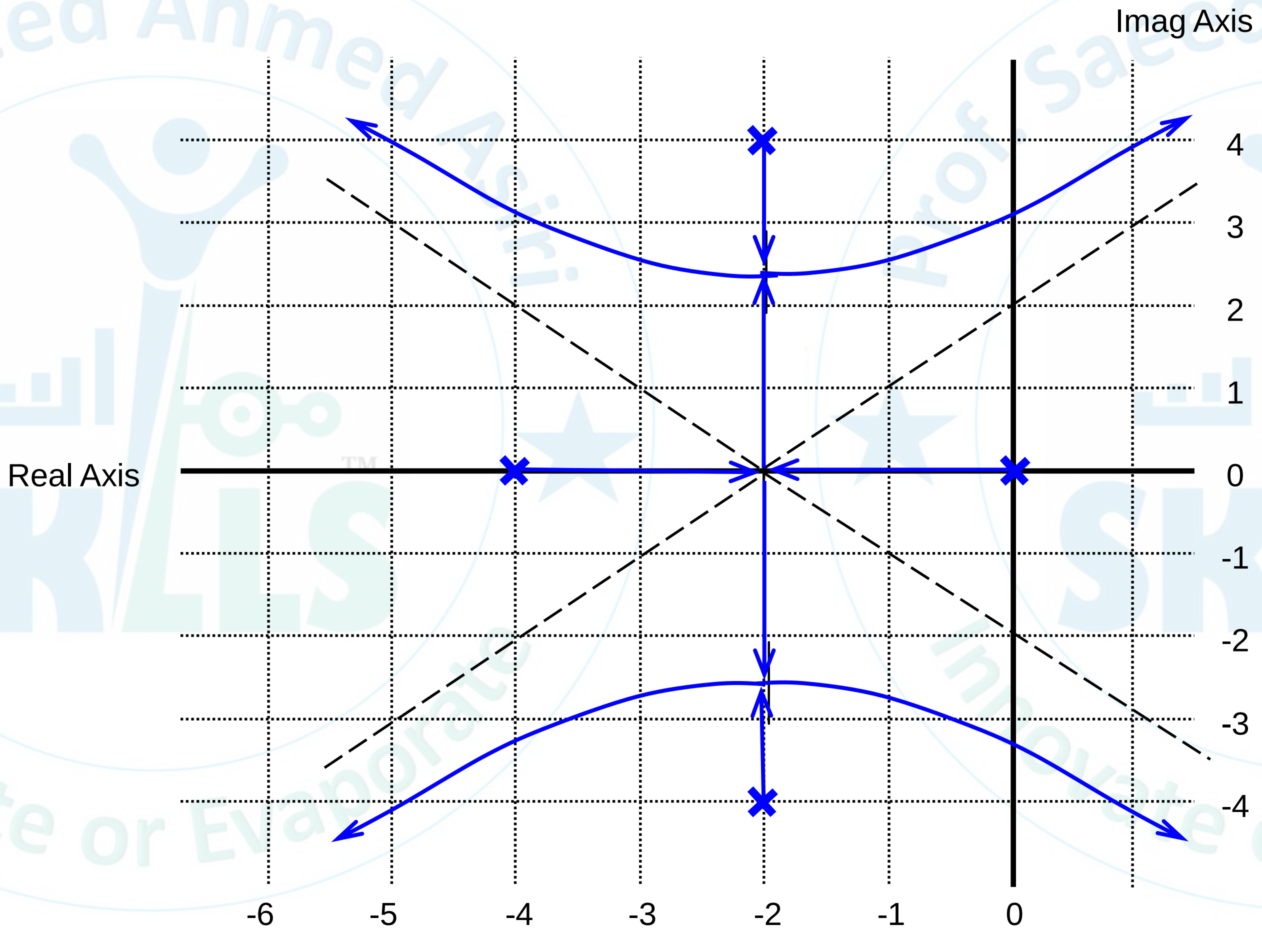
CLCE

$$\xRightarrow{s=j\omega}\left(\omega^4-36\omega^2+K\right)+\left(-8\omega^3+80\omega\right)j=0$$

$$\Rightarrow\begin{cases}\omega^4-36\omega^2+K=0\\ -8\omega^3+80\omega=0\end{cases}\Rightarrow\begin{cases}K_1=0\\ \omega_1=0\end{cases},\begin{cases}K_2=260\\ \omega_2=\sqrt{10}=3.16\end{cases}$$

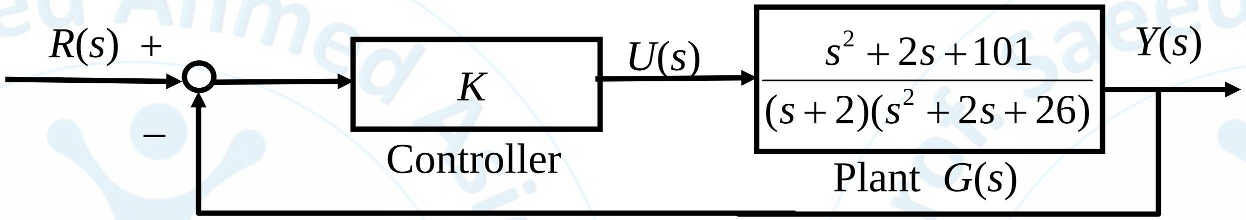
Example 3

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.



Example 4

A feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain *K* varies from 0 to ∞.

Find closed-loop characteristic equation:

$$1+K\frac{s^2+2s+101}{(s+2)(s^2+2s+26)}=0$$

Example 4

Step 1: Formulate the (closed-loop) characteristic equation into the standard form for sketching root locus:

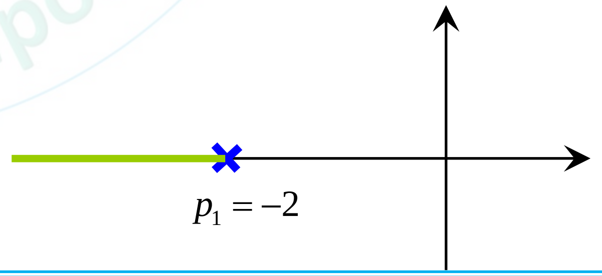
$$1+K\frac{\overbrace{s^2+2s+101}^{N(s)}}{\underbrace{(s+2)(s^2+2s+26)}_{D(s)}}=0$$

Step 2: Find the open-loop zeros, z_i, and the open-loop poles, p_i :

$\boxed{\text{open-loop zeros}}$ $s^2+2s+101=(s+1)^2+100=0,\ z_{1,2}=-1\pm 10j$

$\boxed{\text{open-loop poles}}$ $(s+2)\left((s+1)^2+25\right)=0,\ p_1=-2,\ p_{2,3}=-1\pm 5j$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



Example 4

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

$$N_p - N_z = 1 \qquad \text{One asymptote} \qquad \theta_k = (2k+1)\times 180^\circ = 180^\circ$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

$$z_1 = -1+10j \quad \theta_{z_1} + 90^o - \tan^{-1}(10) - 90^o - 90^o = 180^o$$

$$\theta_{z_1} = 354^o = -6^o$$

$$z_2 = -1-10j \quad \theta_{z_2} = 6^o$$

$$p_1 = -2 \qquad \theta_{p_1} = 180^o$$

$$p_2 = -1+5j \qquad \theta_{p_2} = 11^o$$

$$p_3 = -1-5j \qquad \theta_{p_2} = -11^o$$

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

$$(s+2)(s^2+2s+26) + K(s^2+2s+101) = 0$$

$$\Leftrightarrow s^3 + (4+K)s^2 + (30+2K)s + (52+101K) = 0$$

$$\overset{s=j\omega}{\Rightarrow} \left[(52+101K) - (4+K)\omega^2\right] + \left[(30+2K) - \omega^2\right]\omega j = 0$$

$$\begin{cases} (52+101K) - (4+K)\omega^2 = 0 \\ \left[(30+2K) - \omega^2\right]\omega = 0 \end{cases} \Rightarrow \begin{cases} \omega_1 = 0 \\ K_1 = -\dfrac{52}{101} \end{cases}, \begin{cases} \omega_2 = 9.5 \\ K_2 = 30.4 \end{cases}, \begin{cases} \omega_3 = 5.7 \\ K_3 = 1.1 \end{cases}$$

Example 4

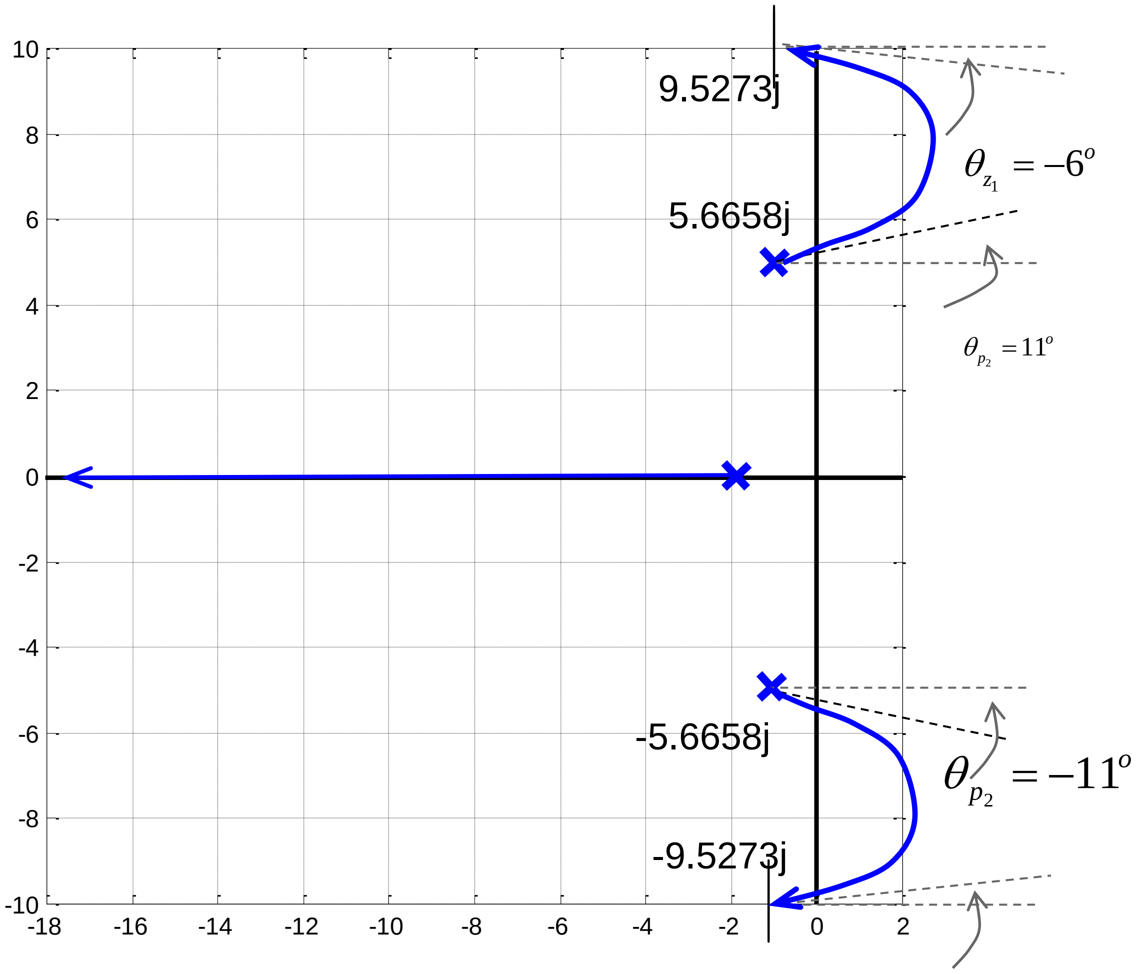
Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.

Stability condition

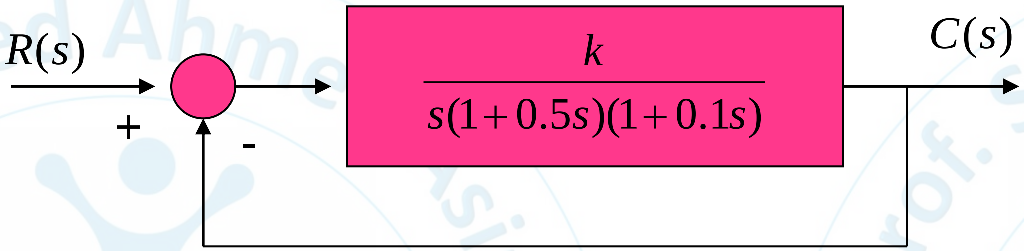
$$0 < K < 1.1$$

or

$$K > 30.4$$



Example 5



$$kGH(s)=\frac{k}{s(1+0.5s)(1+0.1s)}$$

(i) $$poles = 0, \quad -2, \quad -10$$

$$zeros = \infty, \quad \infty, \quad \infty$$

(ii) $$\sigma = \frac{0+(-2)+(-10)-0}{3-0} = -4$$

$$\theta_k = \frac{180}{3-0} = 60$$

$$s(1+0.5s)(1+0.1s)+k=0$$

$$0.05s^3+0.6s^2+s+k=0$$

(iii) $$\frac{dk}{ds} = -\frac{d}{ds}(0.05s^3+0.6s^2+s)=0$$

$$s_1 = -0.945, s_2 = -7.05$$

Example 5

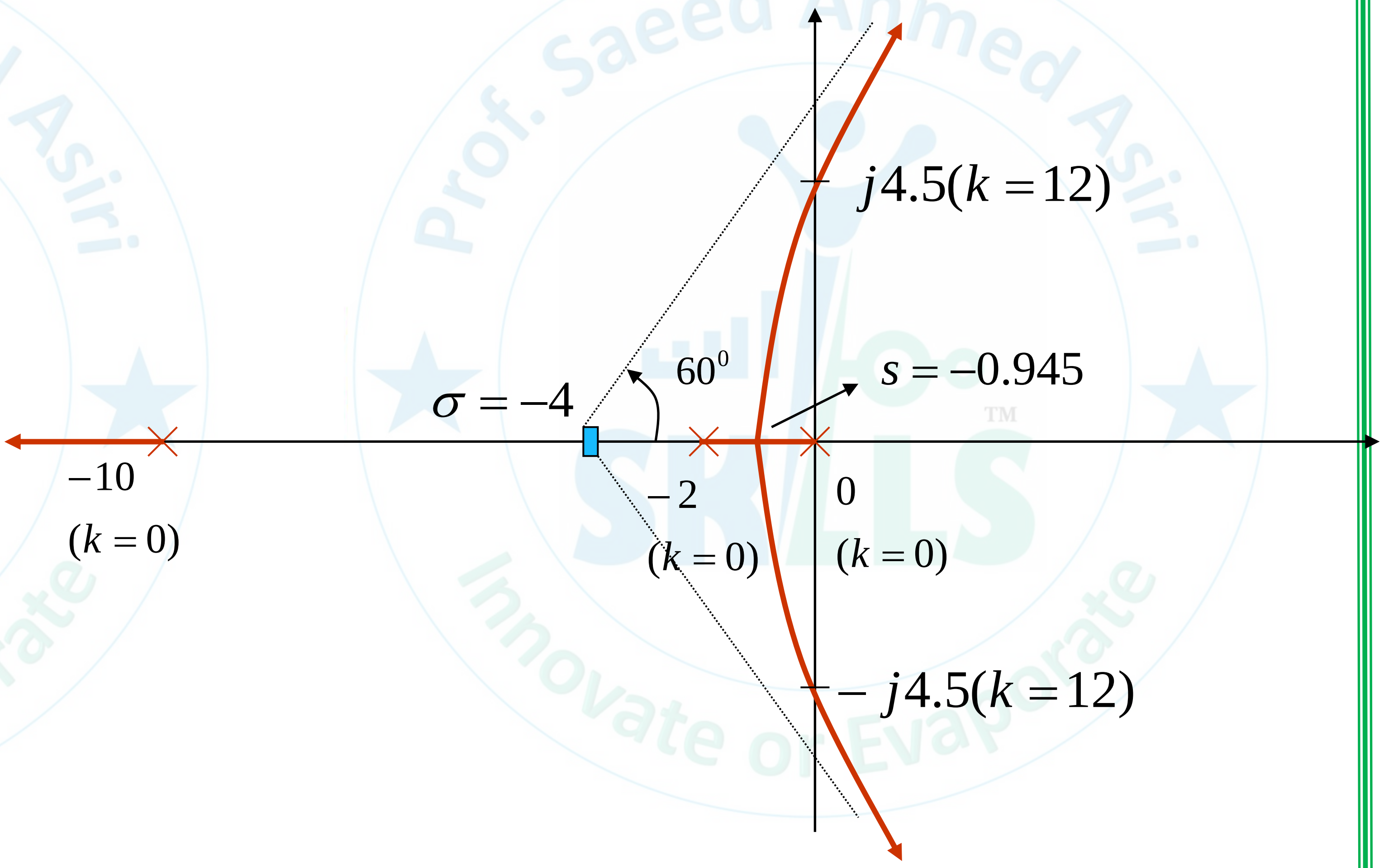
$$0.05s^3 + 0.6s^2 + s + k = 0$$

$$\begin{array}{c|cc} s^3 & 0.05 & 1 \\ s^2 & 0.6 & k \\ s^1 & \dfrac{0.6-0.05k}{0.6} & \\ s & k & \end{array}$$

$$k = 12$$

$$0.6s^2 + 12 = 0$$

$$s = \pm j4.5$$

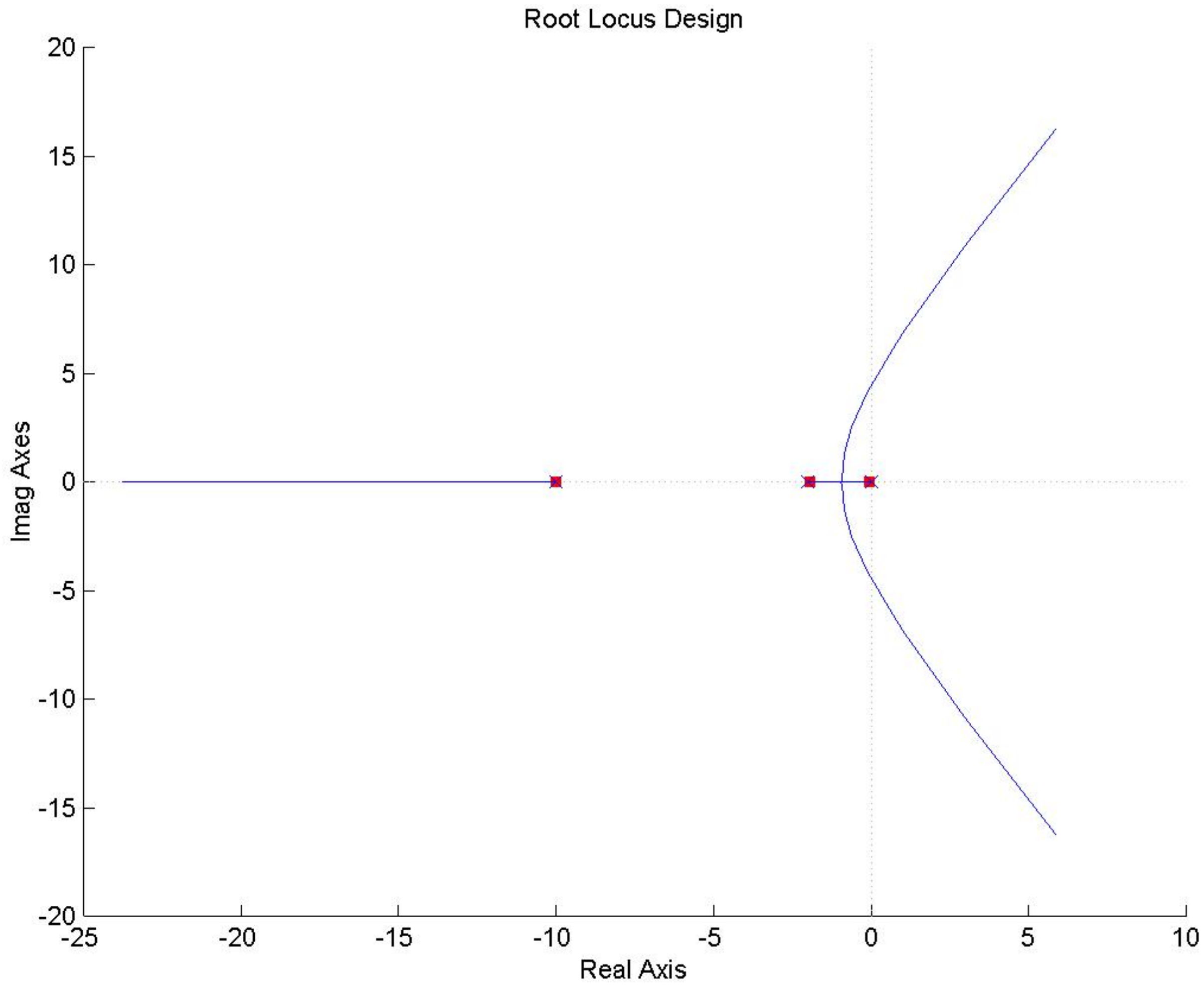


Example 5

$$kGH(s) = \frac{k}{s(1+0.5s)(1+0.1s)}$$

MATLAB method

gh=zpk([],[0 –2 -10],[1])
rltool(gh)



Example 6

$$kGH(s)=\frac{k(-3s-9)}{s^4-s^3-s^2-15s}$$

MATLAB method

```
n=[-3 -9]
m=[1 –1 –1 –15 0]
gh=tf(n,m)
rltool(gh)
```