

# MENG366

# Root Locus Plots

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# Root Locus

- **Motivation**

To satisfy transient performance requirements, it may be necessary to know how to choose certain controller parameters so that the resulting closed-loop poles are in the performance regions, which can be solved with Root Locus technique.

- **Definition**

A graph displaying the roots of a polynomial equation when one of the parameters in the coefficients of the equation changes from 0 to  $\infty$ .

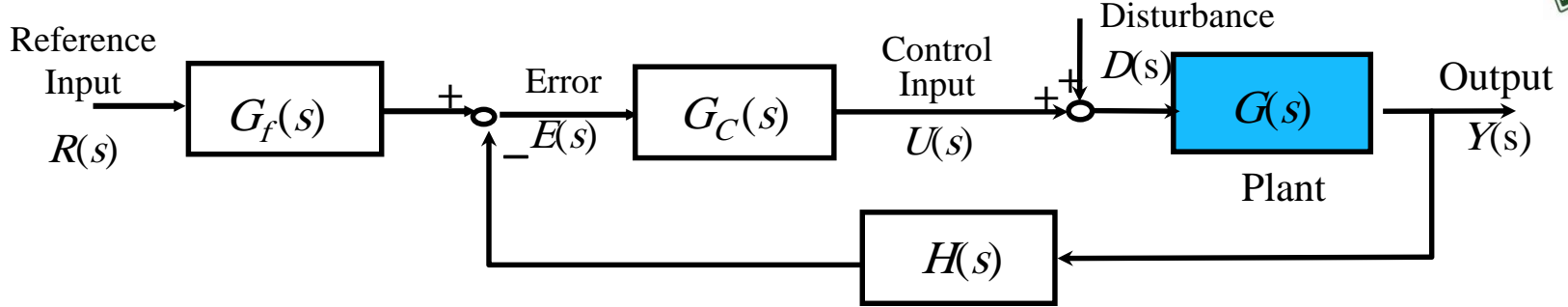
- **Rules for Sketching Root Locus**

- **Examples**

- **Controller Design Using Root Locus**

Letting the CL characteristic equation (CLCE) be the polynomial equation, one can use the Root Locus technique to find how a positive controller design parameter affects the resulting CL poles, from which one can choose a right value for the controller parameter.

# Closed-Loop Characteristic Equation (CLCE)



The closed-loop transfer function  $G_{YR}(s)$  is:

$$G_{YR}(s) = \frac{G(s)G_c(s)G_f(s)}{1 + G(s)G_c(s)H(s)}$$

The closed-loop characteristic equation (CLCE) is:

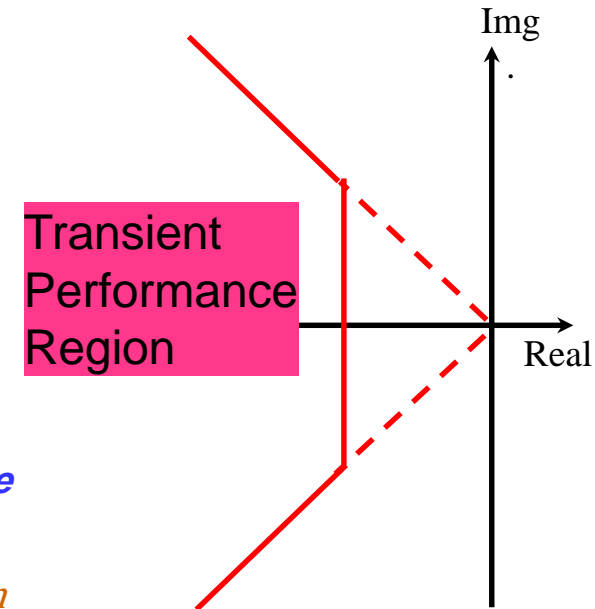
$$1 + G(s)G_c(s)H(s) = 0$$

For simplicity, assume a simple proportional feedback controller:

$$G_c(s) = K_p \Rightarrow 1 + K_p GH = 0$$

*The transient performance specifications define a region on the complex plane where the closed-loop poles should be located.*

*Q: How should we choose  $K_p$  such that the CL poles are within the desired performance boundary?*

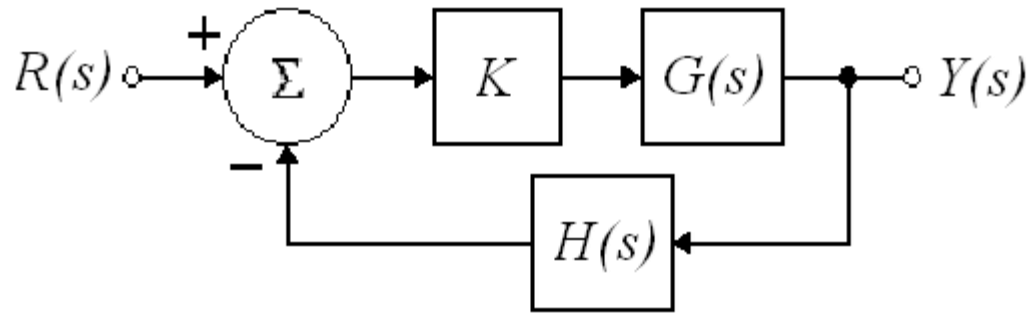


# Goal

Learn a *specific technique* which shows how  
*changes in one of a system's parameter*  
(usually the controller gain,  $K$ )  
*will modify the location of the closed-loop poles*  
in the  $s$ -domain.

# Definition

- The closed-loop poles of the negative feedback control:

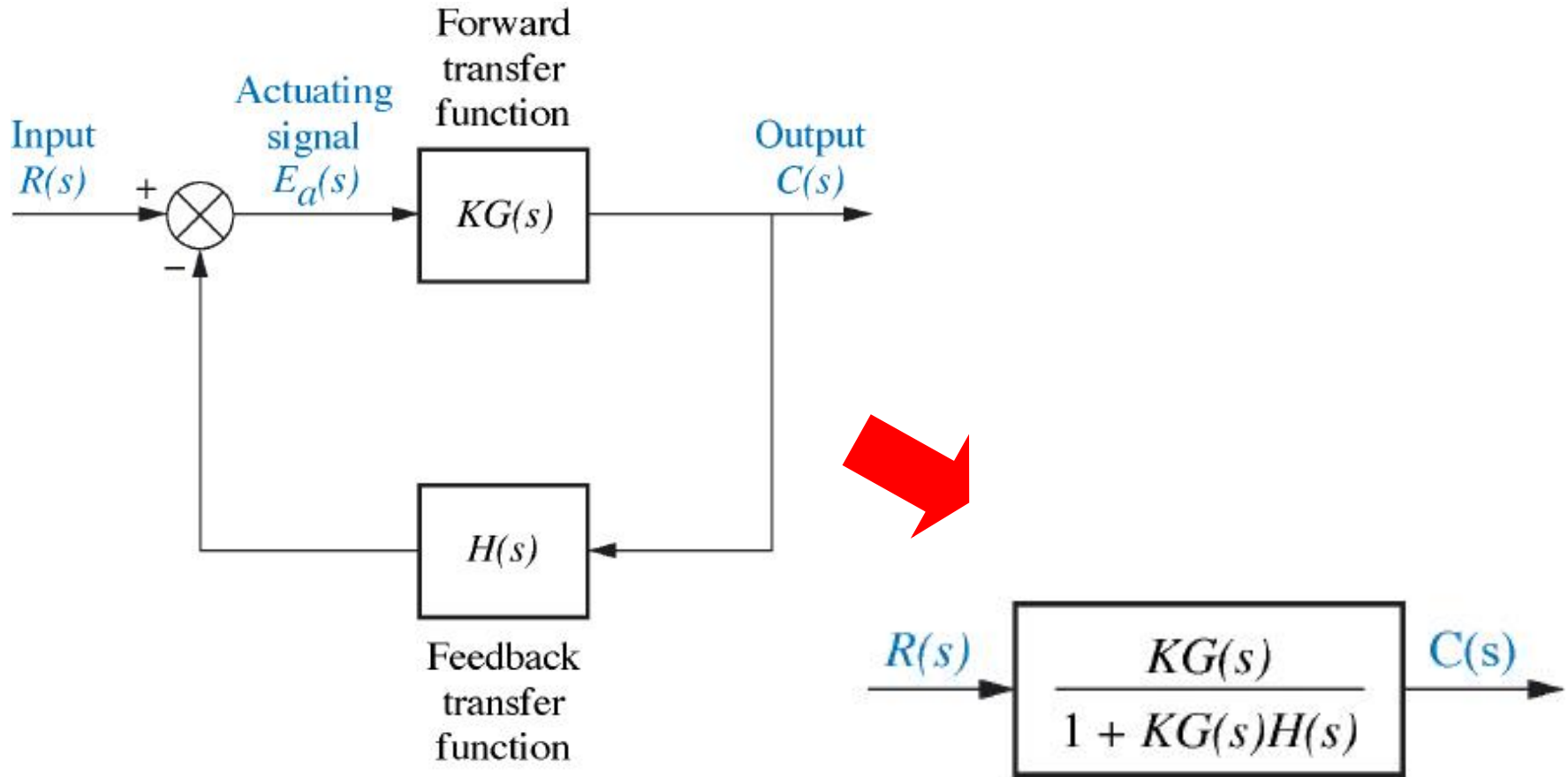


are the roots of the characteristic equation:

$$1 + KG(s)H(s) = 0$$

**The root locus is the locus of the closed-loop poles when a specific parameter (usually gain, K) is varied from 0 to infinity.**

# Introduction: Root Locus



# Introduction: Root Locus

## Characteristic equation

$$1 + KG(s)H(s) = 0$$

$$\Rightarrow 1 + K \frac{N(s)}{D(s)} = 0$$

where  $N(s)$  = numerator of open-loop  $GH$ ,

$D(s)$  = denominator of open-loop  $GH$

Solution of characteristic equation gives poles of closed-loop system

# Introduction: Root Locus

Characteristic eq<sup>n</sup> can be written as:

a.  $D(s) + K N(s) = 0$

**If  $K = 0$**

$$D(s) + K N(s) = 0 \quad \rightarrow \quad D(s) = 0$$

Poles of closed-loop system equal the poles of the open-loop system



# Introduction: Root Locus

Characteristic eq<sup>n</sup> can be written as:

b. 
$$\frac{1}{K} D(s) + N(s) = 0$$

If  $K = \infty$

$$\frac{1}{K} D(s) + N(s) = 0 \quad \rightarrow \quad N(s) = 0$$

Poles of closed-loop system equal the zeros of the open-loop system

# Introduction: Root Locus

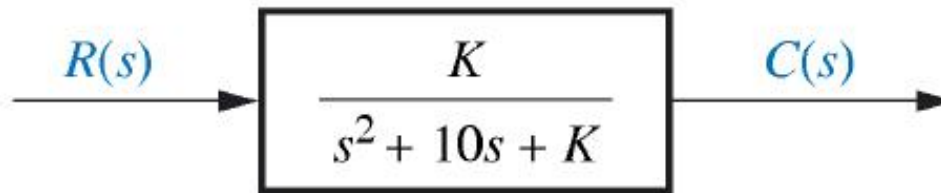
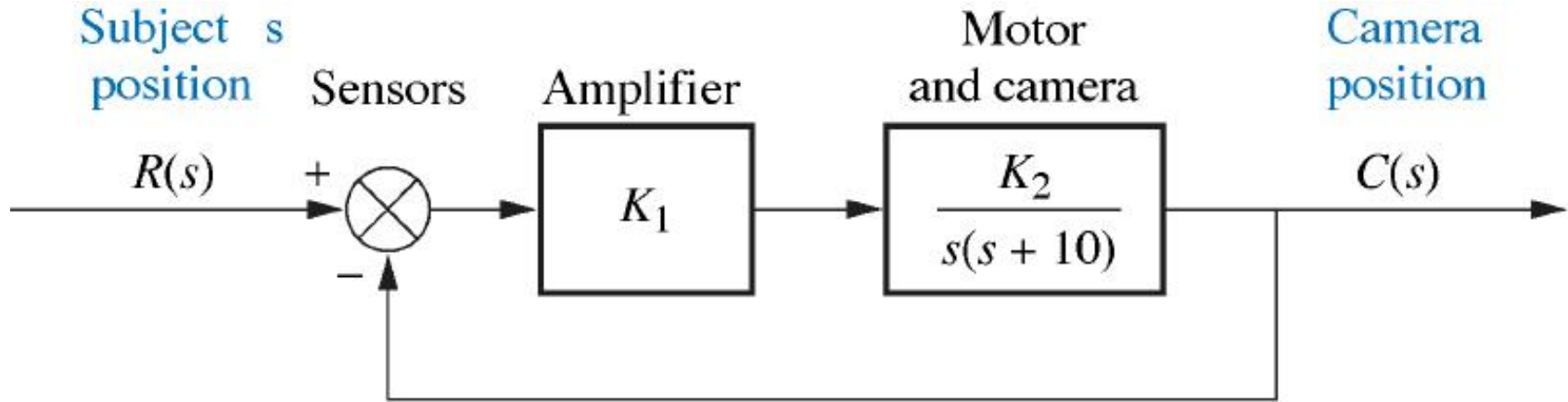
**Poles of closed-loop system are:**

- The poles of the open-loop system when  $K=0$ ,
- The zeros of the open-loop system when  $K=\infty$ .

Root Locus is the locus of the poles of closed-loop system with  $K$  varying from 0 to  $\infty$ .

The root locus starts from the poles of the open-loop system when  $K=0$  and ends at the zeros of the open-loop system when  $K=\infty$ .

# Example: Root Locus



where  $K = K_1 K_2$

# Example: Root Locus

## Characteristic equation

$$s^2 + 10s + K = 0$$

## Rewriting the Characteristic equation as

$$1 + K \frac{N(s)}{D(s)} = 0$$



$$1 + \frac{K}{s^2 + 10s} = 0$$

# Example: Root Locus

$$\rightarrow 1 + \frac{K}{s^2 + 10s} = 0$$

The poles of the open-loop system are:  $s=0$ ,  $s=-10$

There are no zeros of the open-loop system.

The poles of the closed-loop system are:

$$s_{1,2} = -5 \pm \sqrt{25 - K}$$

# Example: Root Locus

The poles of the closed-loop system are:

$$s_{1,2} = -5 \pm \sqrt{25 - K}$$

When  $K=0$

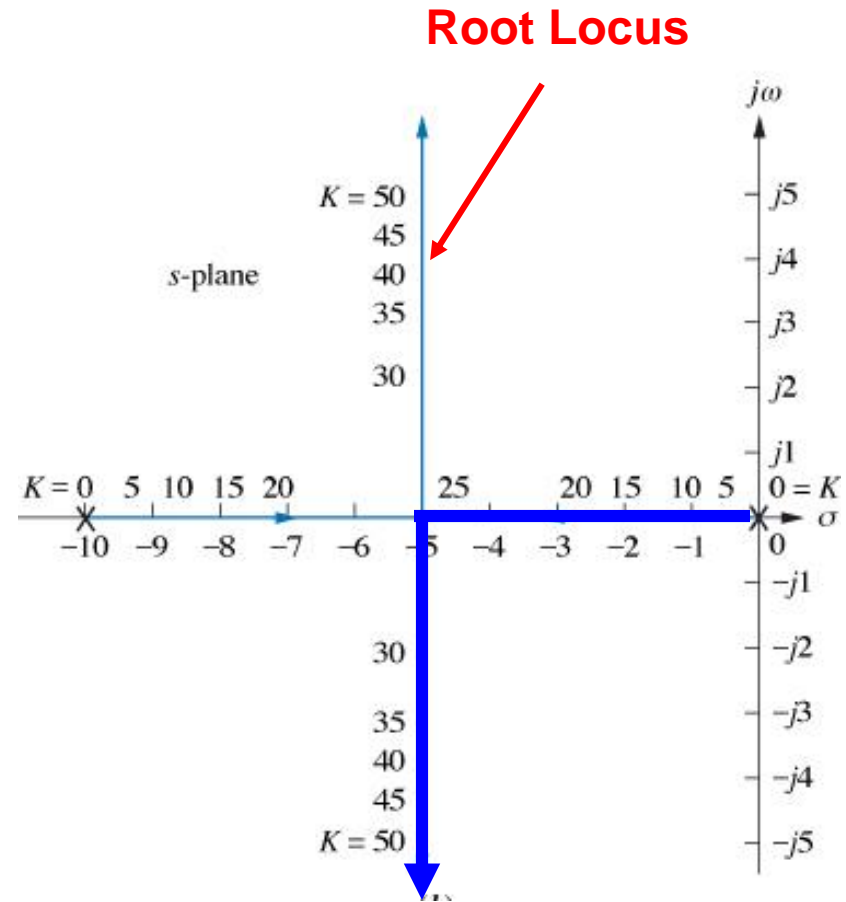
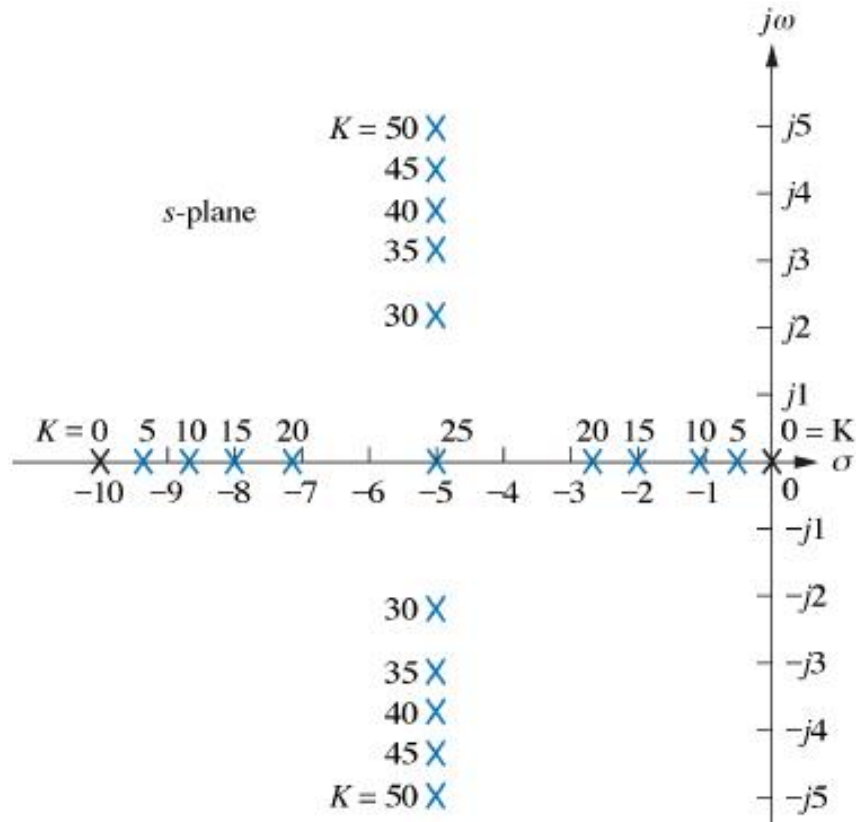
$$s_{1,2} = -5 \pm 5 = 0, -10$$

which are the poles of the open-loop system are:  $s = 0, s = -10$

# Example: Root Locus

$K$	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

# Example: Root Locus



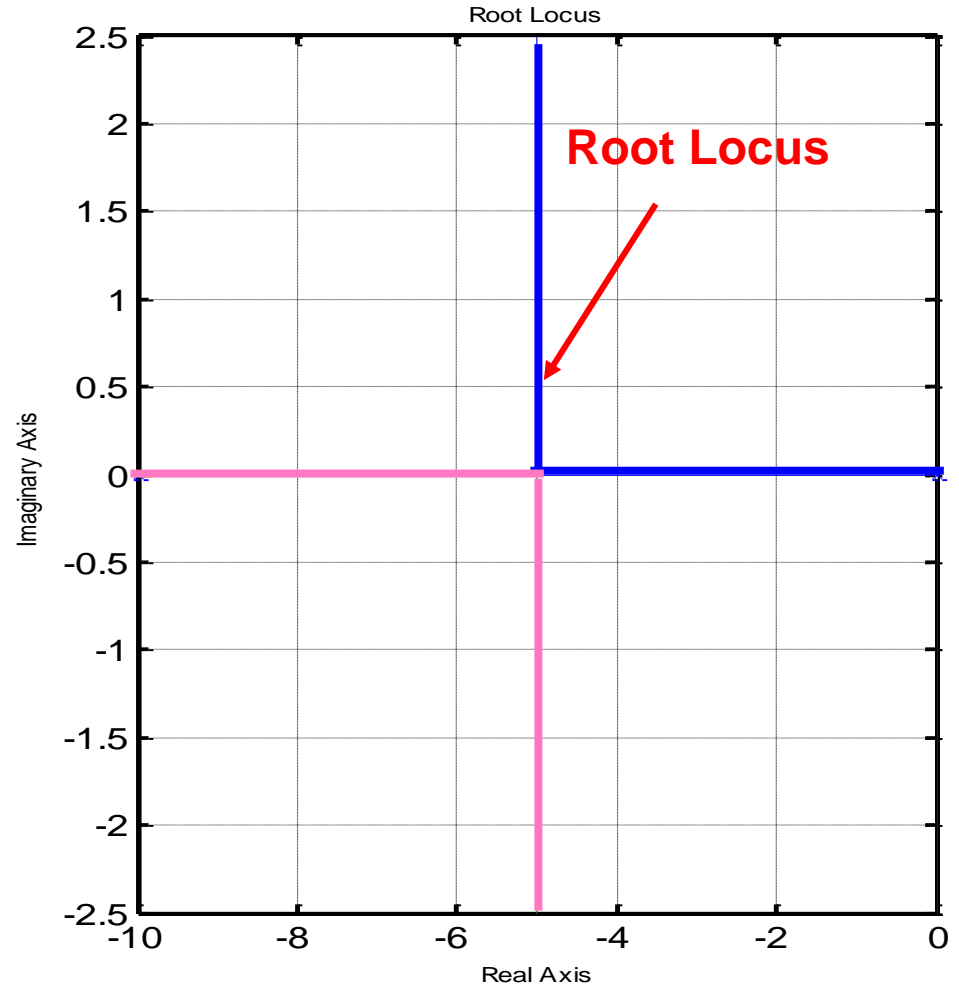


# Example: Root Locus

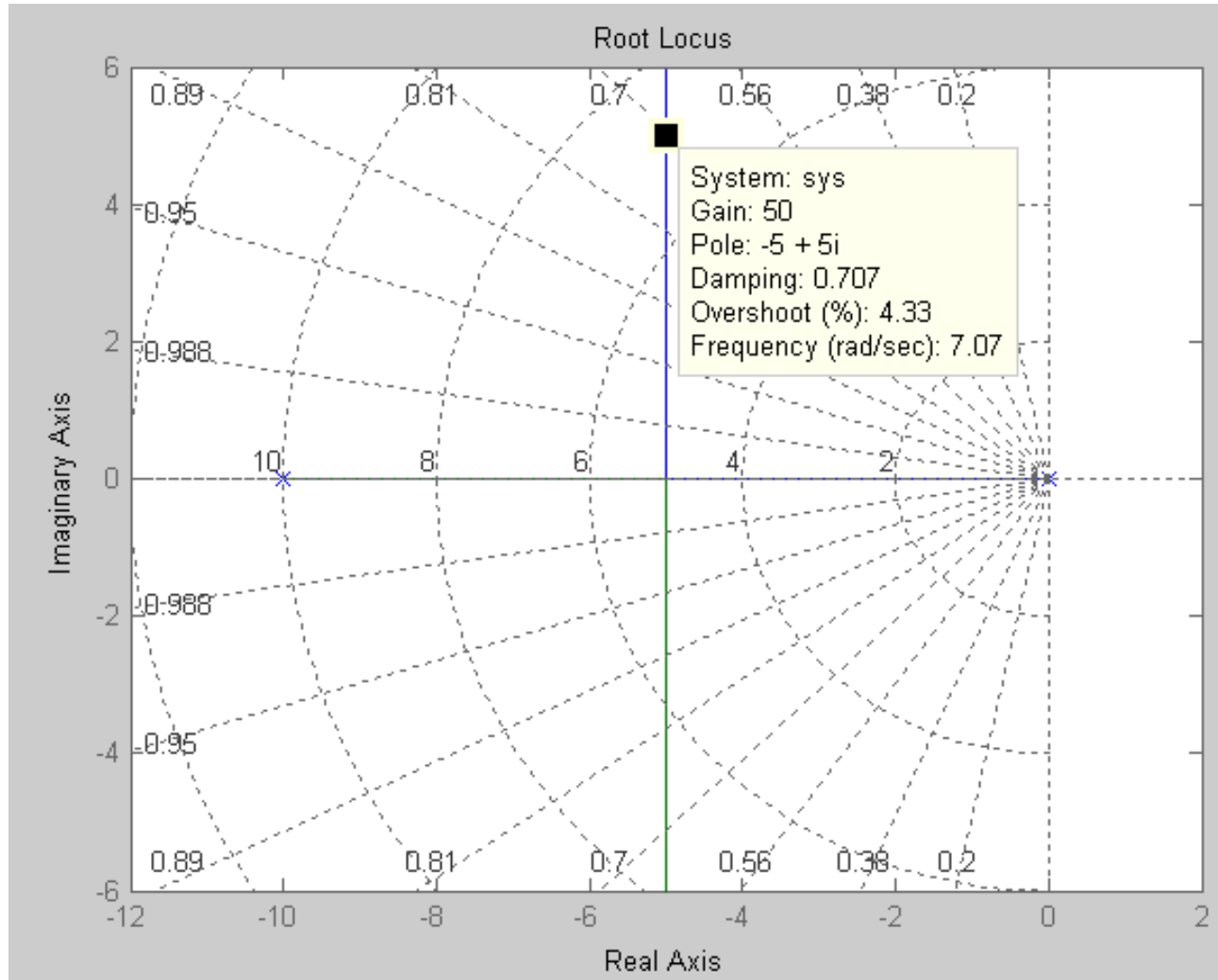
## MATLAB

$$1 + \frac{K}{s^2 + 10s} = 0$$

```
>> n=1;
>> d=[1 10 0];
>> rlocus(n,d)
```



# Example: Root Locus



# Properties of Root Locus

- Root locus has branches equal to number of the open-loop poles.
- Each branch starts from a pole & ends at a zero of the open-loop system.
- If there are no zeros, the root locus branch starts from a pole and seeks a zero at infinity.
- Number of branches going to infinity =  $P-Z$ , where  $P = \#$ of poles &  $Z = \#$ of zeros.

# Root Locus Method Foundations

- The value of  $s$  in the  $s$ -plane that make the loop gain  $KG(s)H(s)$  equal to  $-1$  are the closed-loop poles

$$(i.e. \quad 1 + KG(s)H(s) = 0 \Leftrightarrow KG(s)H(s) = -1 \quad )$$

- $KG(s)H(s) = -1$  can be split into two equations by equating the magnitudes and angles of both sides of the equation.

# Properties of Root Locus

As  $1 + KG(s)H(s) = 0$



a. **Magnitude Property**  $|KG(s)H(s)| = 1,$

**or**  $K = \frac{1}{|G(s)H(s)|}$

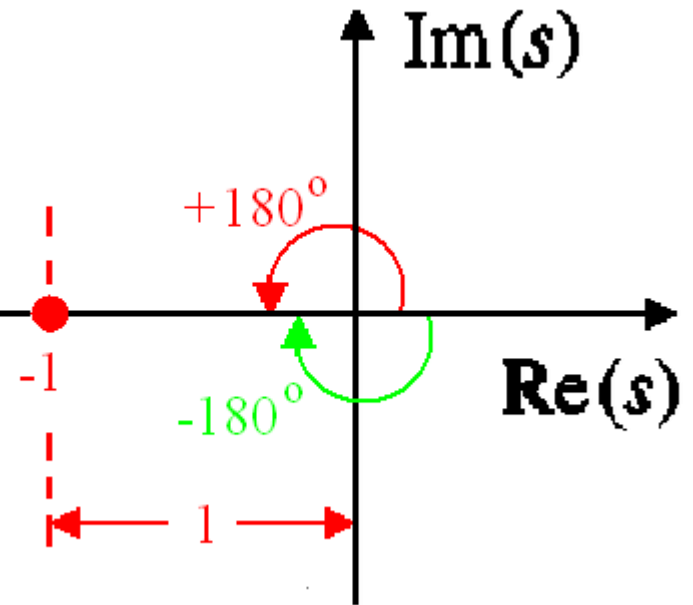
b. **Phase angle Property**

$$\angle [KG(s)H(s)] = (2k + 1)\pi \quad k = 0, \pm 1, \pm 2, \dots$$

# Angle and Magnitude Conditions

$$KG(s)H(s) = -1$$

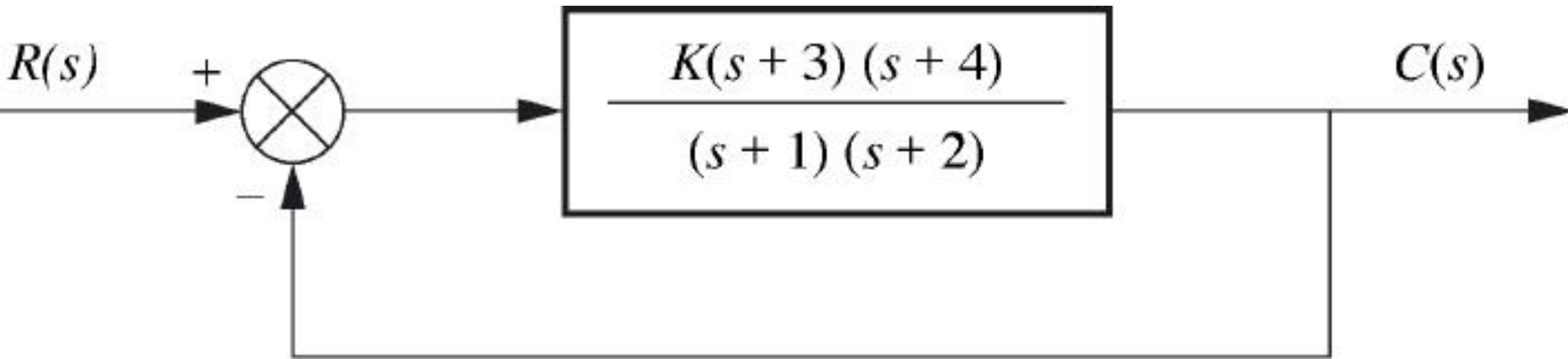
$$\Leftrightarrow \begin{cases} |KG(s)H(s)| = 1 \\ \angle KG(s)H(s) = \pm 180^\circ (2l + 1) \\ l = 0, 1, 2, \dots \end{cases}$$



$$\Leftrightarrow \begin{cases} |G(s)H(s)| = 1/K \\ \angle G(s)H(s) = \pm 180^\circ (2l + 1) \quad l = 0, 1, 2, \dots \end{cases}$$

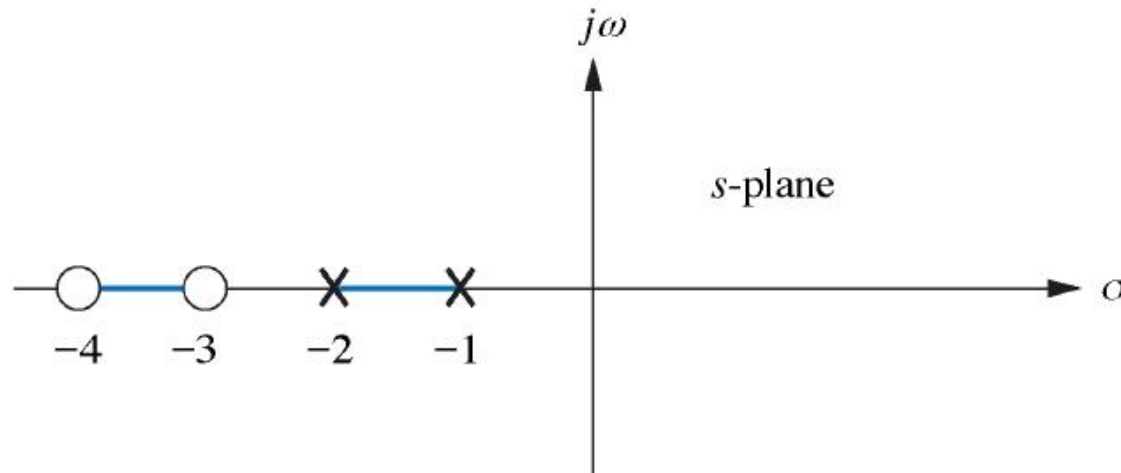
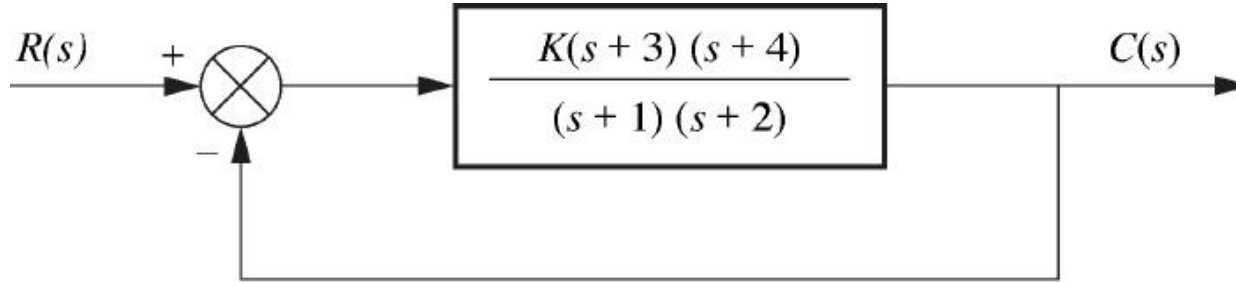
Independent of K

# Example



(a)

# Example



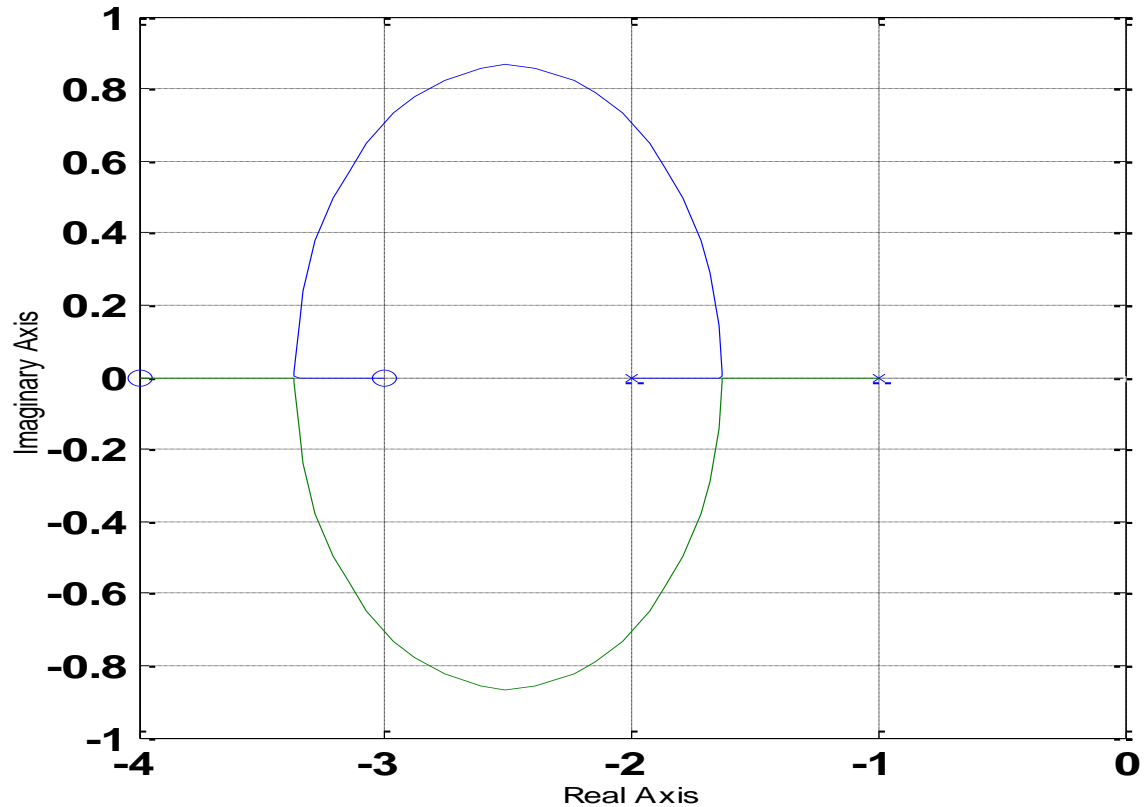


# Example

## MATLAB

$$KGH = \frac{K(s+3)(s+4)}{(s+1)(s+2)}$$

```
>> n=[1 7 12];
>> d=[1 3 2];
>> rlocus(n,d)
```



# Example

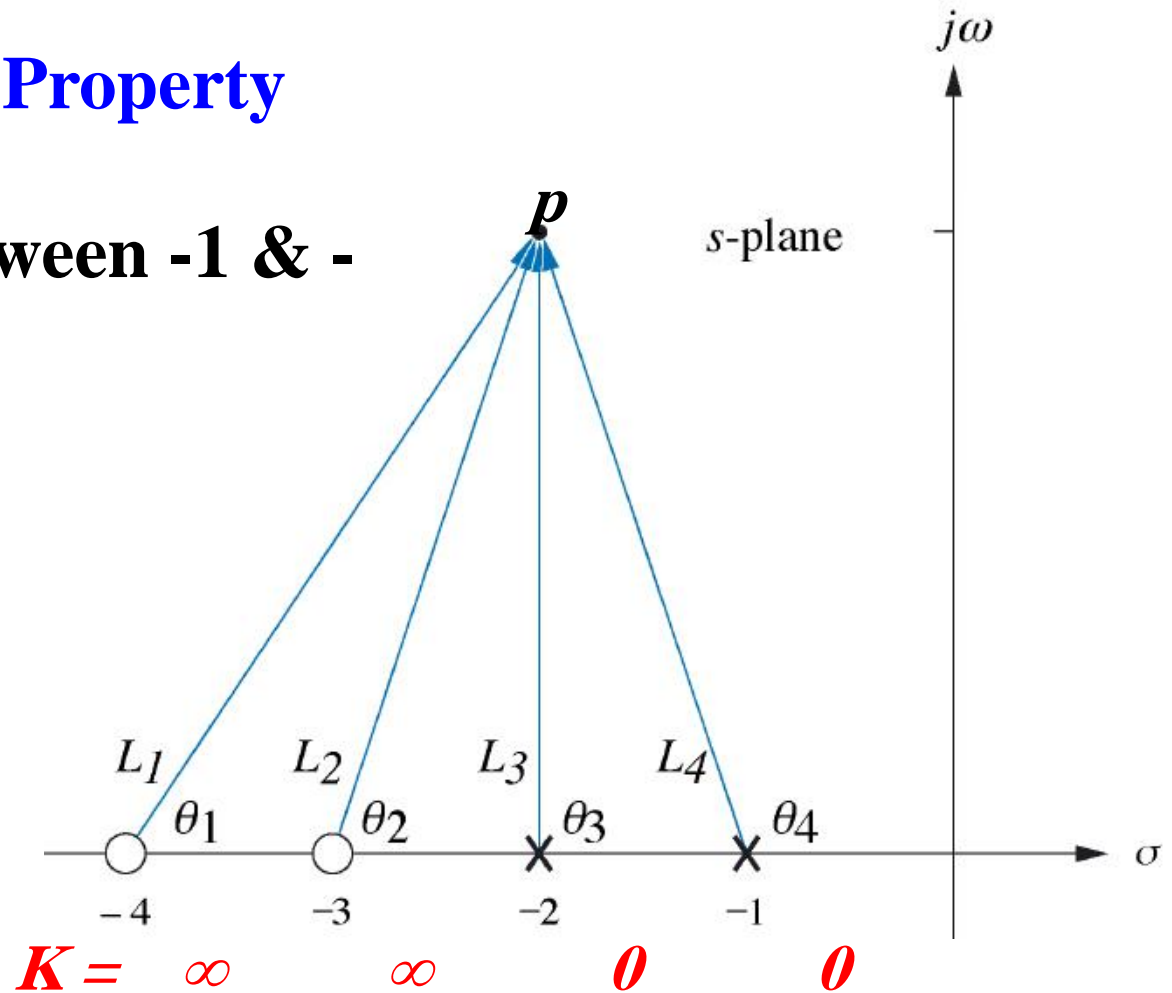
## Check Magnitude Property

If  $p = -1.5$  (lies between -1 & -2)

$$K = \frac{1}{|G(s)H(s)|}$$

$$= \frac{1}{\left| \frac{(-1.5 + 3)(-1.5 + 4)}{(-1.5 + 1)(-1.5 + 2)} \right|}$$

$$= \frac{0.25}{1.5 \times 2.5} = 0.0667$$

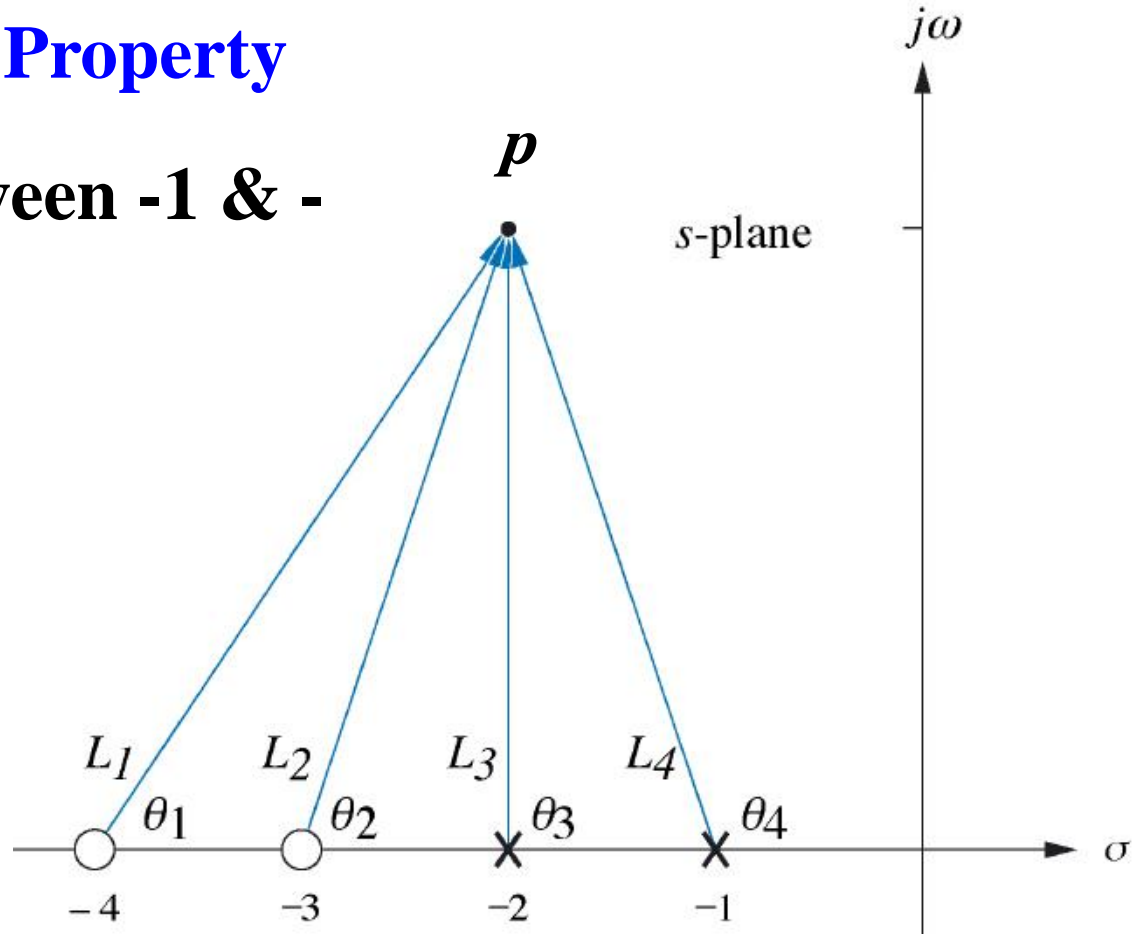


# Example

## Check Phase angle Property

If  $p = -1.5$  (lies between -1 & -2)

$$\angle [KG(s)H(s)] = \frac{\angle(s+3) + \angle(s+4)}{\angle(s+1) + \angle(s+2)} = \frac{0 + 0}{180 + 0} = -180^\circ$$



# Asymptotes of Root Locus

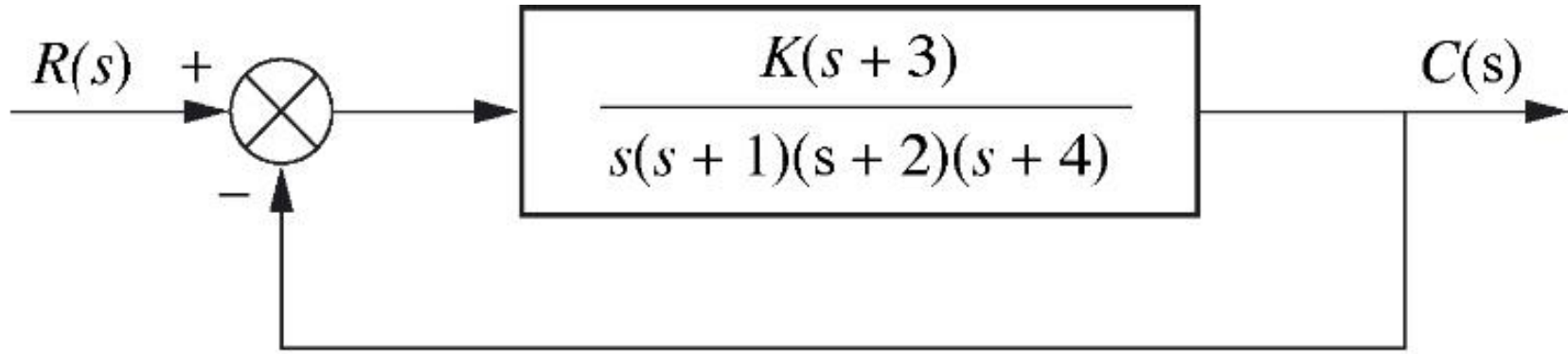
The branches of the root locus have asymptotes such that

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}}$$

$$\theta_a = \frac{(2k + 1)\pi}{\# \text{finite poles} - \# \text{finite zeros}},$$

For  $k = 0, \pm 1, \pm 2, \dots$

# Asymptotes of Root Locus

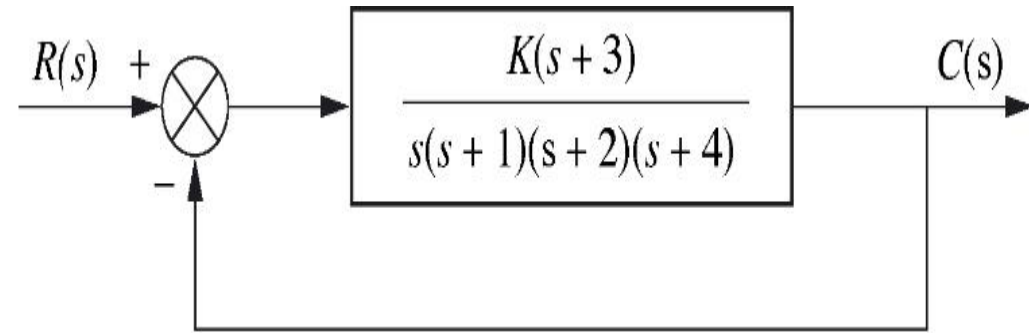
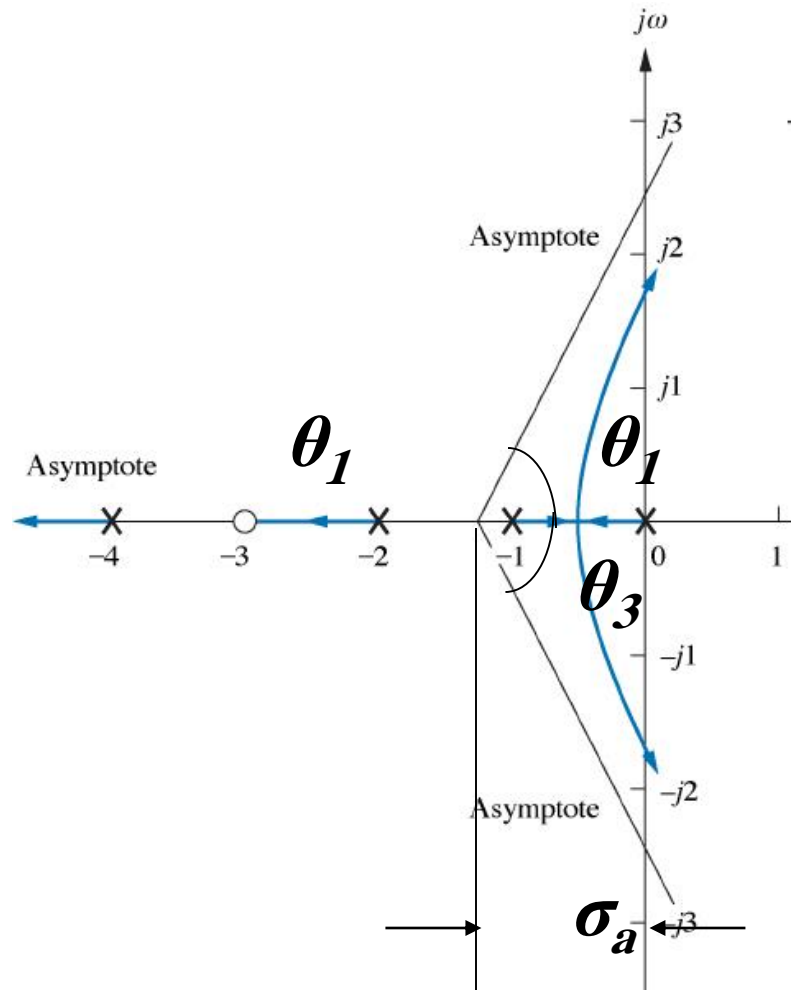


$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}}$$

$$\theta_a = \frac{(2k + 1)\pi}{\# \text{finite poles} - \# \text{finite zeros}},$$

For  $k = 0, \pm 1, \pm 2, \dots$

# Asymptotes of Root Locus



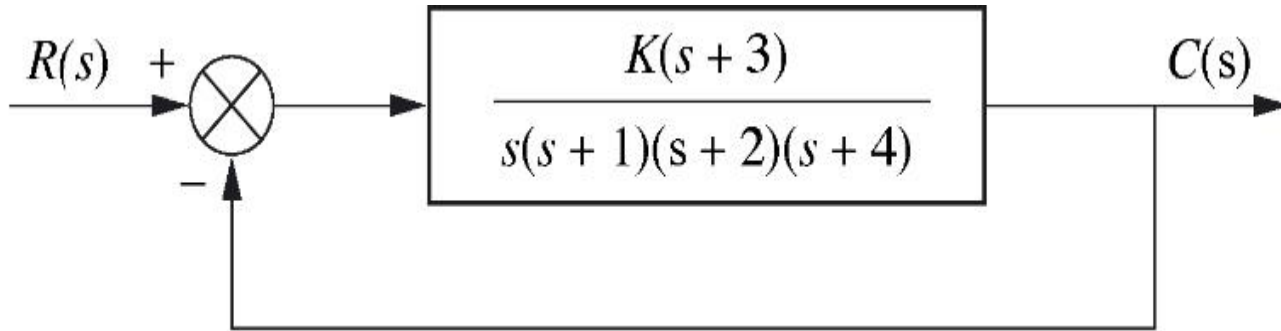
$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}}$$

$$= \frac{-7 + 3}{4 - 1} = -\frac{4}{3}$$

$$\theta_a = \frac{(2k+1)\pi}{3}, \text{ For } k = 0, \pm 1, \pm 2, \dots$$

$$= \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

# Imaginary axis Crossing



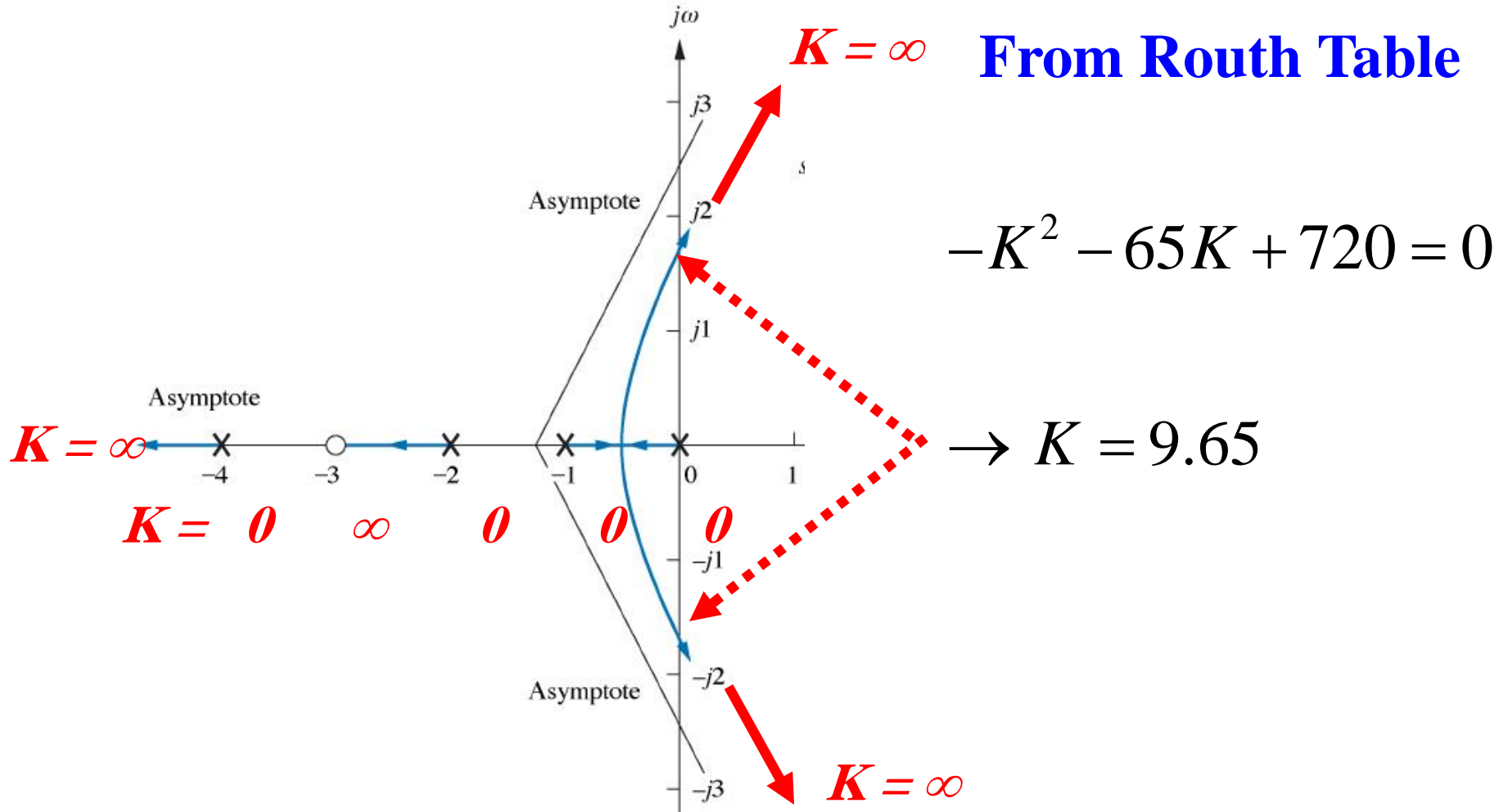
$$T = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$$

---

$s^4$	1	14	$3K$
$s^3$	7	$8 + K$	
$s^2$	$90 - K$	$21K$	
$s^1$	$\frac{-K^2 - 65K + 720}{90 - K}$		
$s^0$	$21K$		

---

# Imaginary axis Crossing

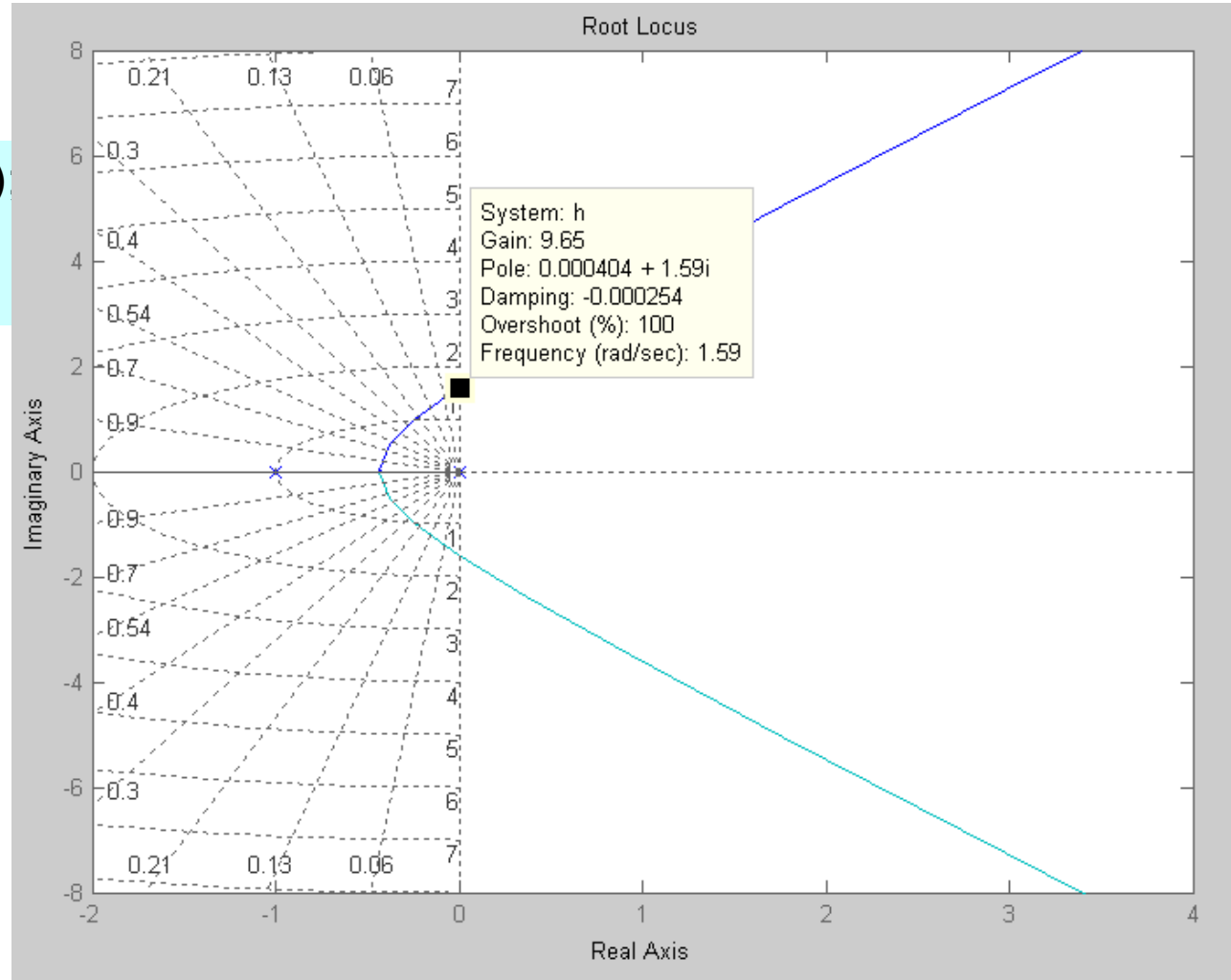


fig\_08\_12

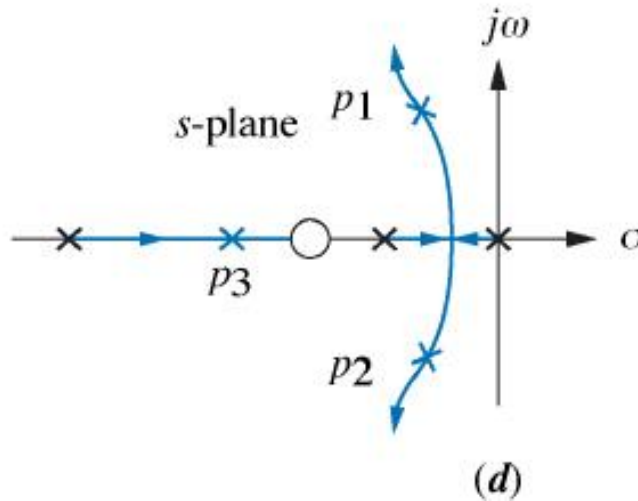
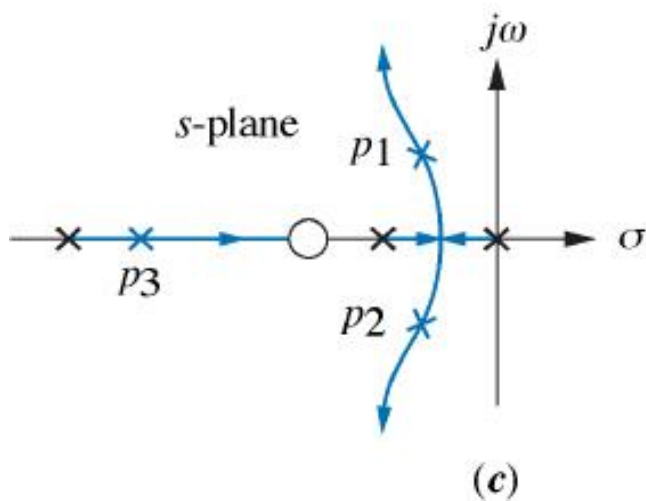
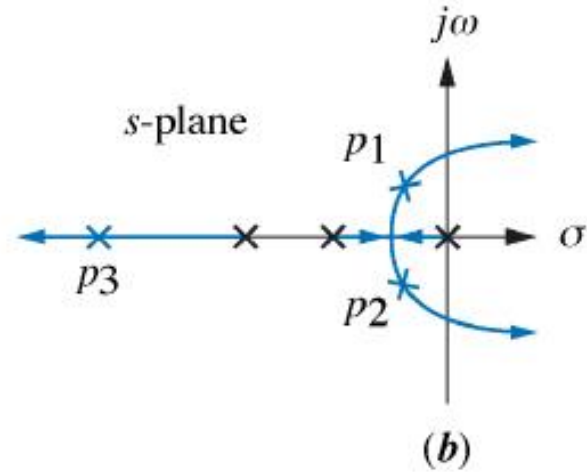
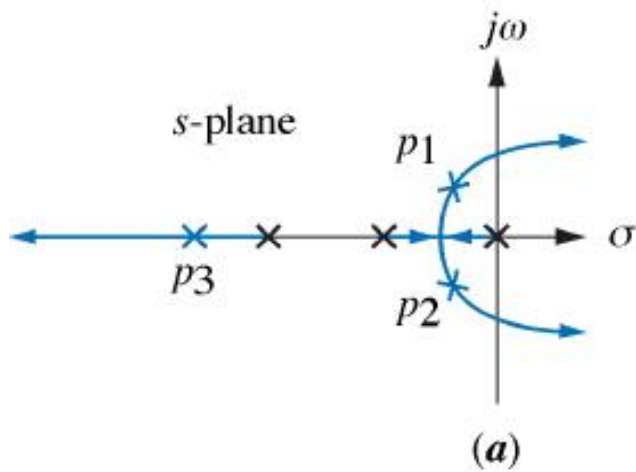


# Imaginary axis Crossing

```
G=zpk([-3],[0 -1 -2 -4],1)
h=tf(G);
rlocus(h)
```



# Transient Response Design



- × Open-loop pole
- × Closed-loop pole
- Closed-loop zero

# Methods of Obtaining Root Locus

- Given a value of  $K$ , numerically solve the  $1 + K G(s) = 0$  equation to obtain all roots. Repeat this procedure for a set of  $K$  values that span from 0 to  $\infty$  and plot the corresponding roots on the complex plane.
- In MATLAB, use the commands *rlocus* and *rlocfind*. A very efficient root locus design tool is the command *rltool*. You can use on-line help to find the usage for these commands.

$$1 + K_P \cdot 0.03 \cdot \frac{16}{s(0.0174s + 1)} = 0 \quad \Rightarrow \quad 1 + K_P \cdot \frac{0.48}{0.0174s^2 + s} = 0$$

No open-loop zeros

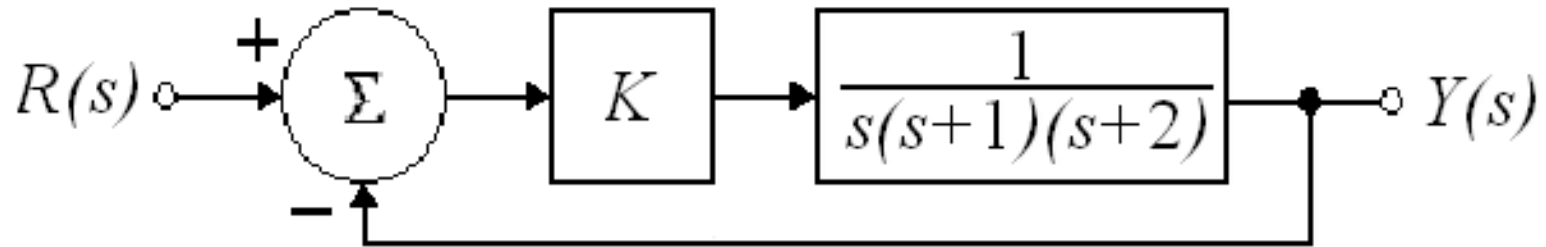
Two open-loop poles

- Apply the following root locus sketching rules to obtain an *approximated* root locus plot.

```
>> op_num=[0.48];
>> op_den=[0.0174 1 0];
>> rlocus(op_num,op_den);
>> [K, poles]=rlocfind(op_num,op_den);
```

# Learning by doing – Example 1

- 1) Sketch the root locus of the following system:



- 2) Determine the value of  $K$  such that the damping ratio  $\zeta$  of a pair of dominant complex conjugate closed-loop is 0.5.

# Rule #1

Assuming  $n$  poles and  $m$  zeros for  $G(s)H(s)$ :

- The  $n$  branches of the root locus start at the  $n$  poles.
- $m$  of these  $n$  branches end on the  $m$  zeros
- The  $n-m$  other branches terminate at infinity along asymptotes.

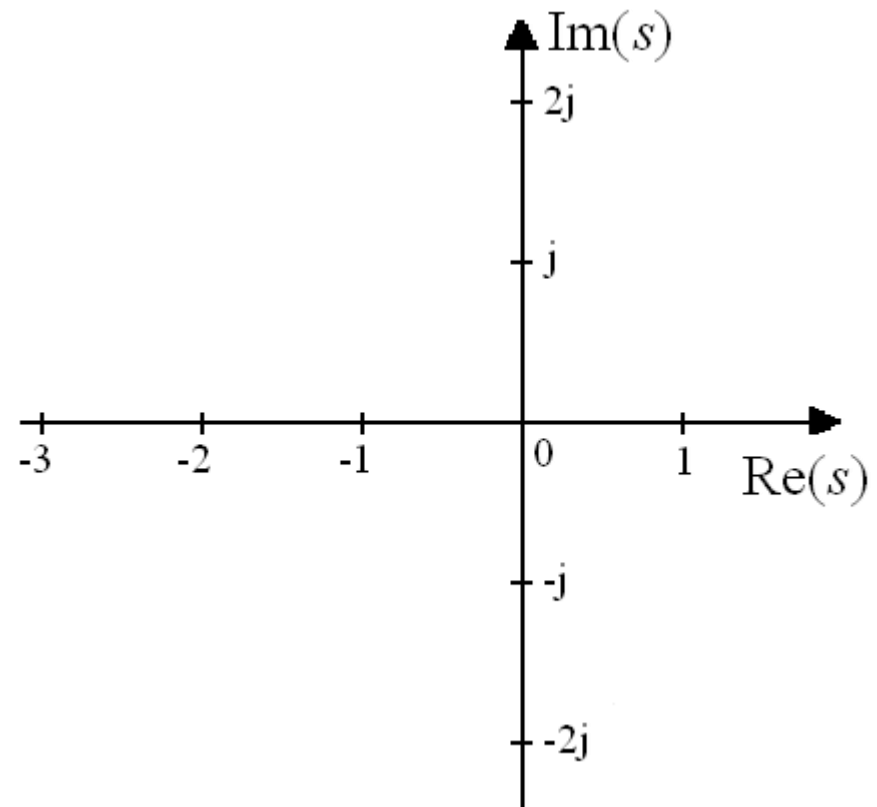
First step: Draw the  $n$  poles and  $m$  zeros of  $G(s)H(s)$  using  $x$  and  $o$  respectively

# Applying Step #1

Draw the  $n$  poles and  $m$  zeros of  $G(s)H(s)$  using  $x$  and  $o$  respectively.

$$G(s)H(s) = \frac{1}{s(s+1)(s+2)}$$

- 3 poles:  
 $p_1 = 0$ ;  $p_2 = -1$ ;  $p_3 = -2$
- No zeros

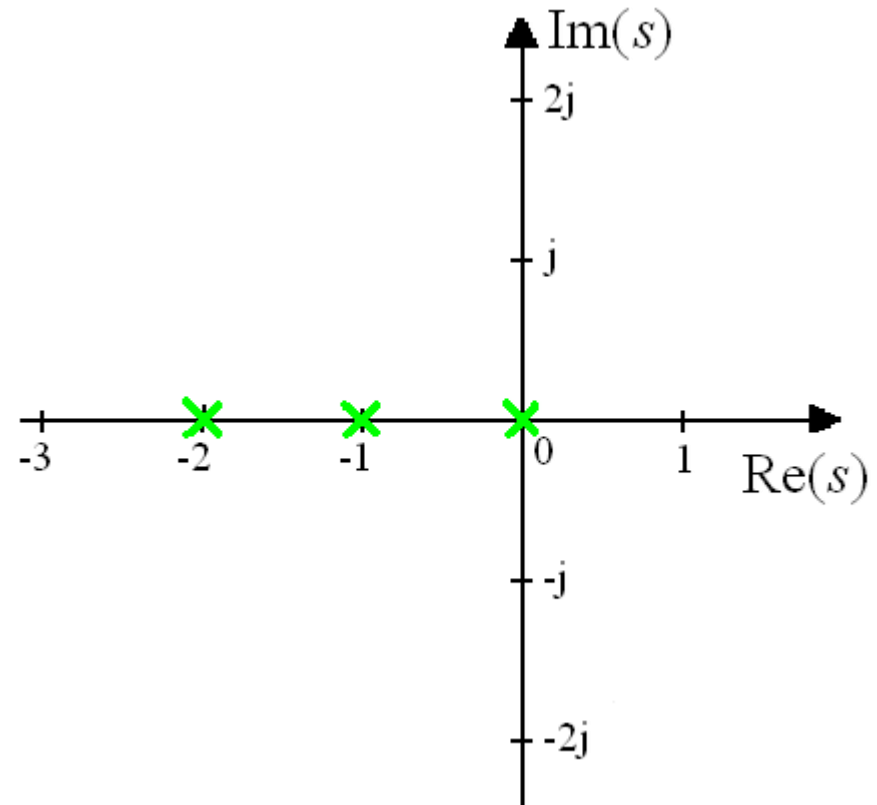


# Applying Step #1

Draw the  $n$  poles and  $m$  zeros of  $G(s)H(s)$  using  $x$  and  $o$  respectively.

$$G(s)H(s) = \frac{1}{s(s+1)(s+2)}$$

- 3 poles:  
 $p_1 = 0$ ;  $p_2 = -1$ ;  $p_3 = -2$
- No zeros



## Rule #2

- The loci on the real axis are to the left of an ODD number of REAL poles and REAL zeros of  $G(s)H(s)$

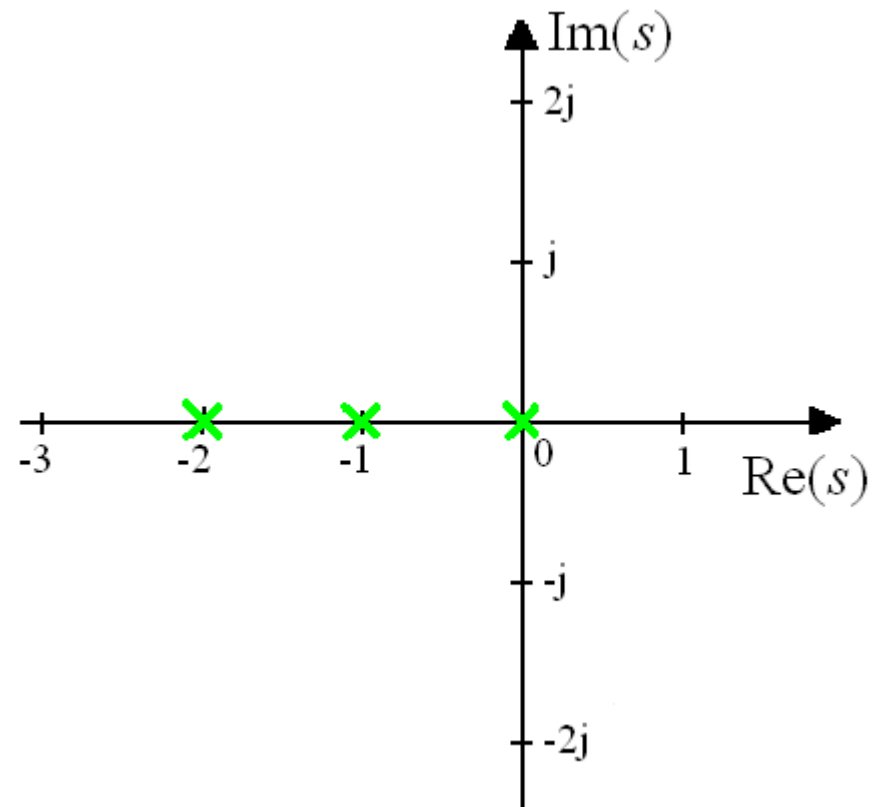
Second step: Determine the loci on the real axis. Choose a arbitrary test point. If the TOTAL number of both real poles and zeros is to the RIGHT of this point is ODD, then this point is on the root locus



# Applying Step #2

Determine the loci on the real axis:

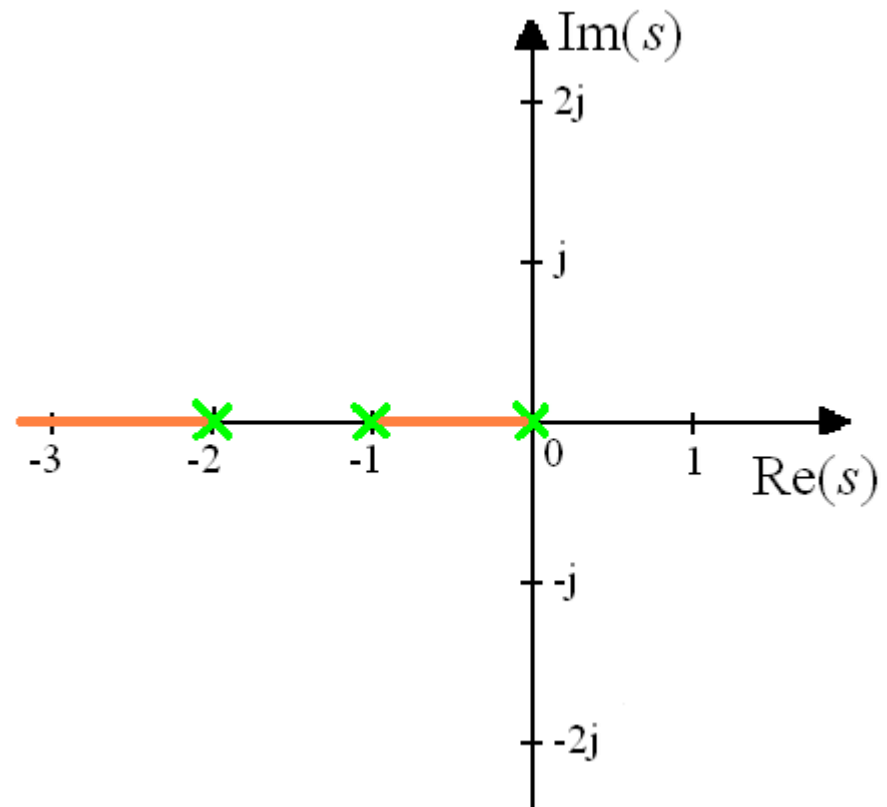
- Choose an arbitrary test point.
- If the TOTAL number of both real poles and zeros is to the RIGHT of this point is ODD, then this point is on the root locus



# Applying Step #2

Determine the loci on the real axis:

- Choose an arbitrary test point.
- If the TOTAL number of both real poles and zeros is to the RIGHT of this point is ODD, then this point is on the root locus



## Rule #3

Assuming  $n$  poles and  $m$  zeros for  $G(s)H(s)$ :

- The root loci for very large values of  $s$  must be asymptotic to straight lines originate on the real axis at point:

$$s = \alpha = \frac{\sum p_i - \sum z_i}{n - m}$$

radiating out from this point at angles:

$$\phi_l = \frac{\pm 180^\circ (2l + 1)}{n - m}$$

Third step: Determine the  $n - m$  asymptotes of the root loci. Locate  $s = \alpha$  on the real axis. Compute and draw angles. Draw the asymptotes using dash lines.

# Applying Step #3

Determine the  $n - m$  asymptotes:

- Locate  $s = \alpha$  on the real axis:

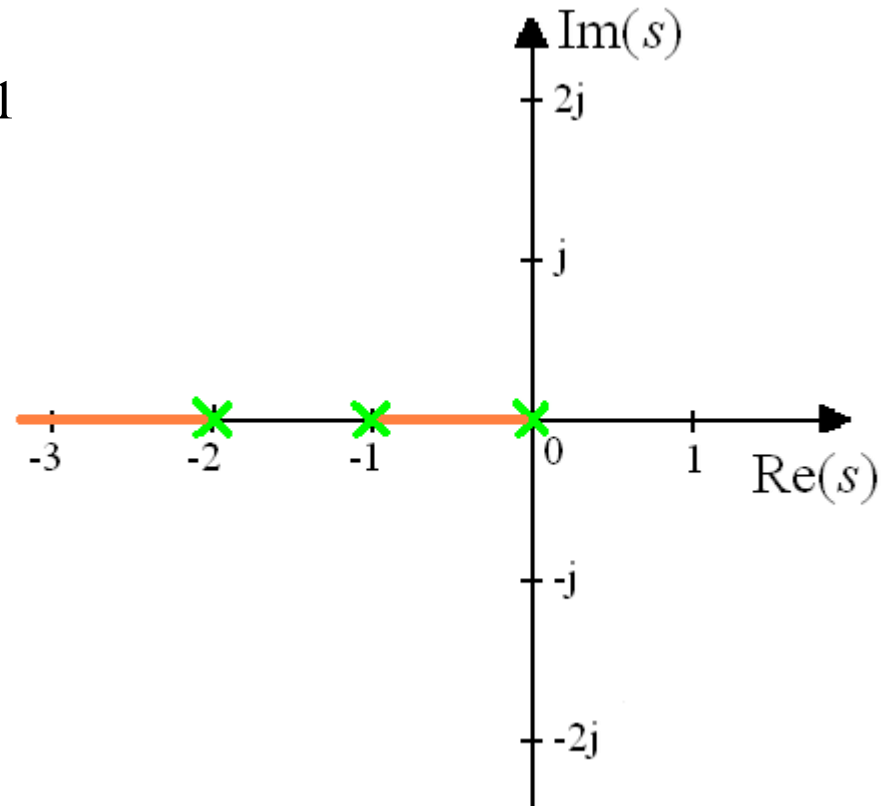
$$s = \alpha = \frac{p_1 + p_2 + p_3}{3 - 0} = \frac{0 - 1 - 2}{3} = -1$$

- Compute and draw angles:

$$\phi_l = \frac{\pm 180(2l + 1)}{n - m} \quad l = 0, 1, 2, \dots$$

$$\Rightarrow \begin{cases} \phi_0 = \frac{\pm 180^0(2 \times 0 + 1)}{3 - 0} = \pm 60^0 \\ \phi_1 = \frac{\pm 180^0(2 \times 1 + 1)}{3 - 0} = \pm 180^0 \end{cases}$$

- Draw the asymptotes using dash lines.



# Applying Step #3

Determine the  $n - m$  asymptotes:

- Locate  $s = \alpha$  on the real axis:

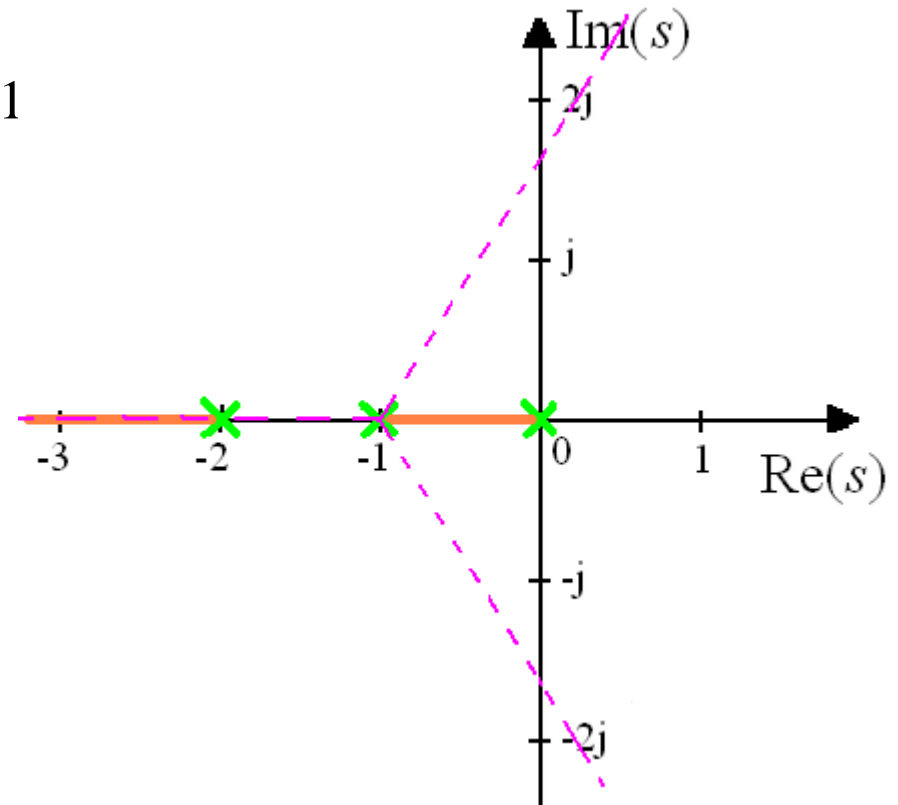
$$s = \alpha = \frac{p_1 + p_2 + p_3}{3 - 0} = \frac{0 - 1 - 2}{3} = -1$$

- Compute and draw angles:

$$\phi_l = \frac{\pm 180(2l + 1)}{n - m} \quad l = 0, 1, 2, \dots$$

$$\Rightarrow \begin{cases} \phi_0 = \frac{\pm 180^0(2 \times 0 + 1)}{3 - 0} = \pm 60^0 \\ \phi_1 = \frac{\pm 180^0(2 \times 1 + 1)}{3 - 0} = \pm 180^0 \end{cases}$$

- Draw the asymptotes using dash lines.



# Breakpoint Definition

- The breakpoints are the points in the  $s$ -domain where **multiple** roots of the characteristic equation of the feedback control occur.
- These points correspond to intersection points on the root locus.

## Rule #4

Given the characteristic equation is  $KG(s)H(s) = -1$

- The **breakpoints** are the closed-loop poles that satisfy:

$$\frac{dK}{ds} = 0$$

Fourth step: Find the breakpoints. Express  $K$  such as:

$$K = \frac{-1}{G(s)H(s)}.$$

Set  $dK/ds = 0$  and solve for the poles.

# Applying Step #4

Find the breakpoints.

- Express  $K$  such as:

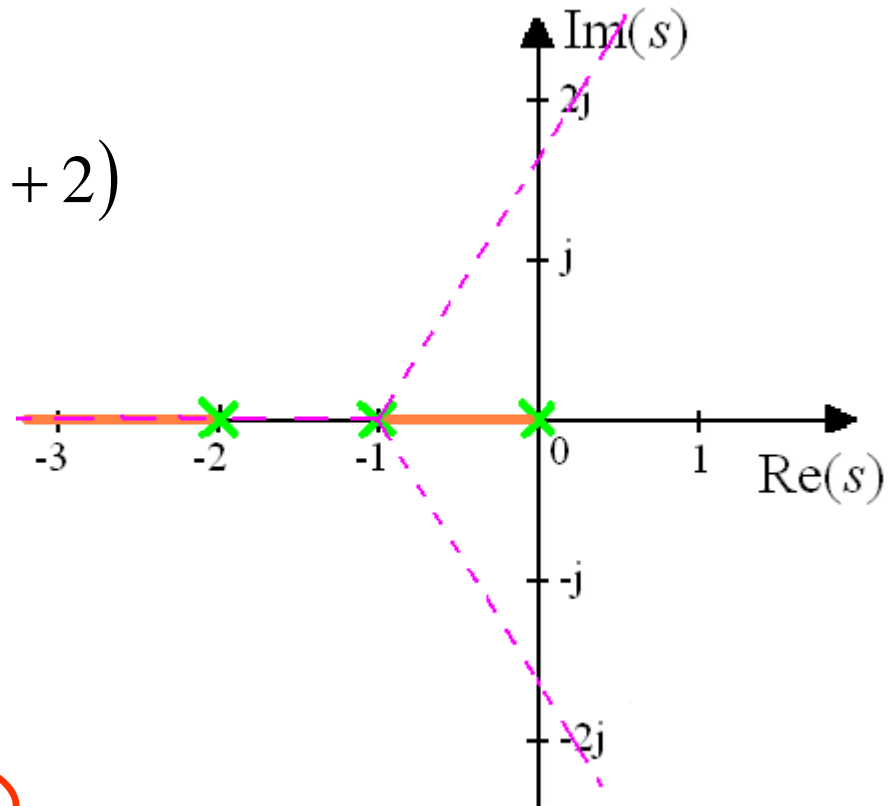
$$K = \frac{-1}{G(s)H(s)} = -s(s+1)(s+2)$$

$$K = -s^3 - 3s^2 - 2s$$

- Set  $dK/ds = 0$  and solve for the poles.

$$-3s^2 - 6s - 2 = 0$$

$$s_1 = -1.5774, \quad s_2 = -0.4226$$





# Applying Step #4

Find the breakpoints.

- Express  $K$  such as:

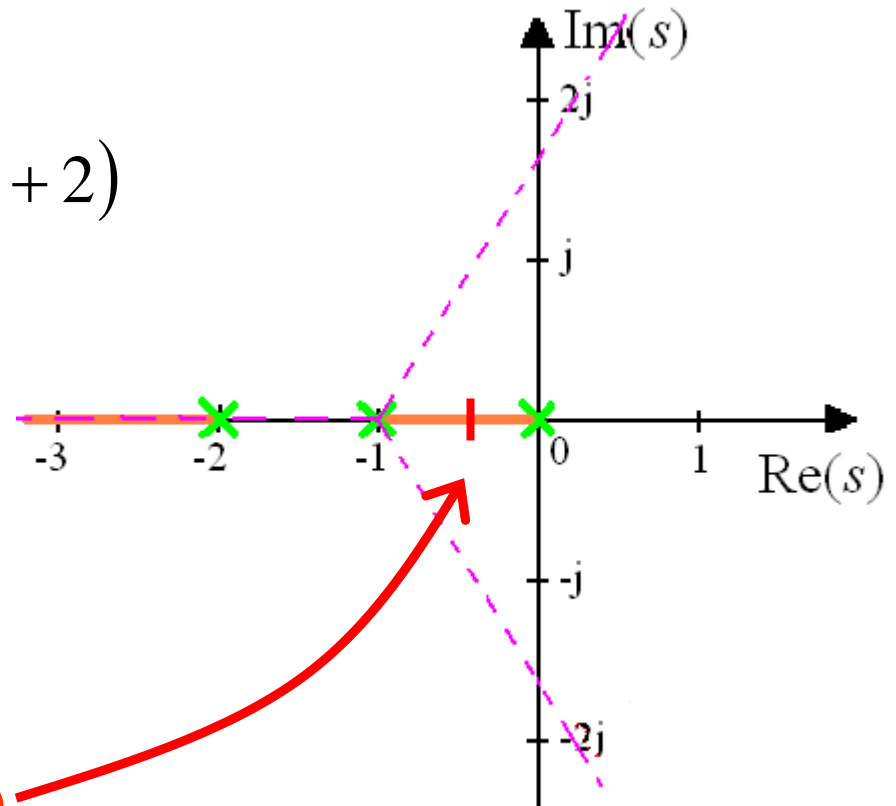
$$K = \frac{-1}{G(s)H(s)} = -s(s+1)(s+2)$$

$$K = -s^3 - 3s^2 - 2s$$

- Set  $dK/ds = 0$  and solve for the poles.

$$-3s^2 - 6s - 2 = 0$$

$$s_1 = -1.5774, \quad s_2 = -0.4226$$



# Recall Rule #1

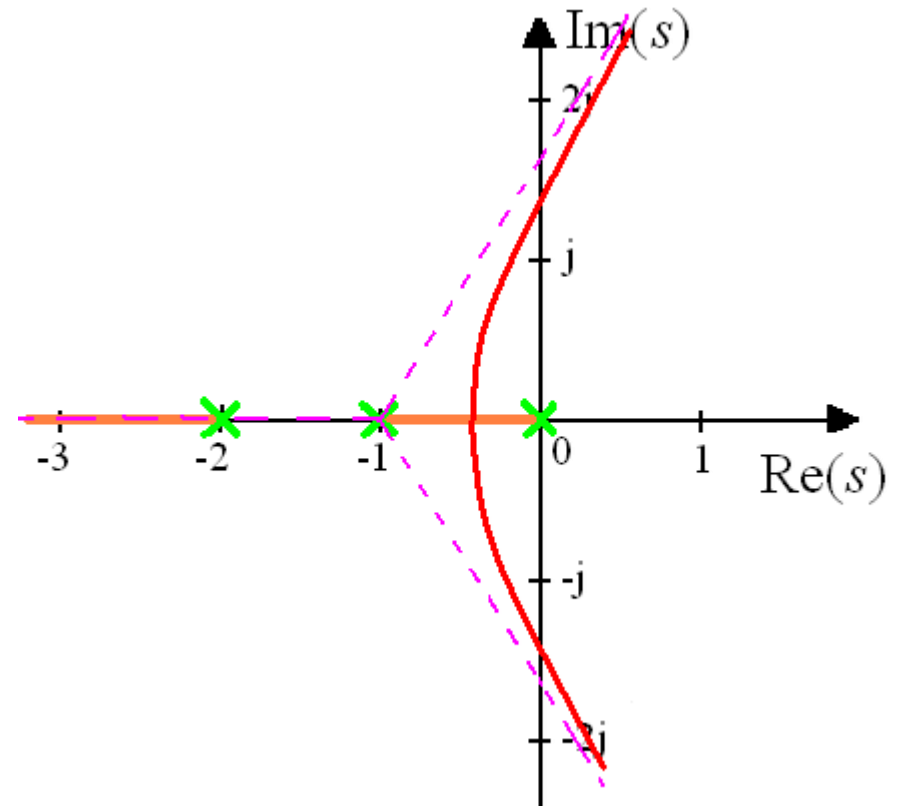
Assuming  $n$  poles and  $m$  zeros for  $G(s)H(s)$ :

- The  $n$  branches of the root locus start at the  $n$  poles.
- $m$  of these  $n$  branches end on the  $m$  zeros
- The  $n-m$  other branches terminate at infinity along asymptotes.

Last step: Draw the  $n-m$  branches that terminate at infinity along asymptotes

# Applying Last Step

Draw the  $n-m$  branches that terminate at infinity along asymptotes



## Points on both root locus & imaginary axis?

- Points on imaginary axis satisfy:

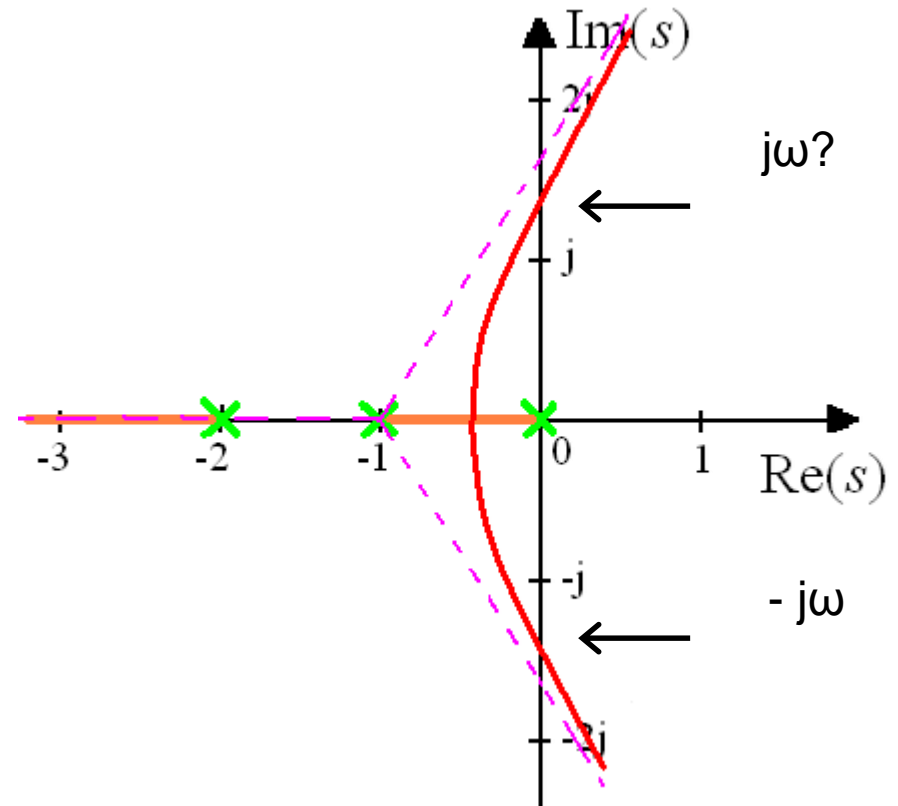
$$s = j\omega$$

- Points on root locus satisfy:

$$1 + KG(s)H(s) = 0$$

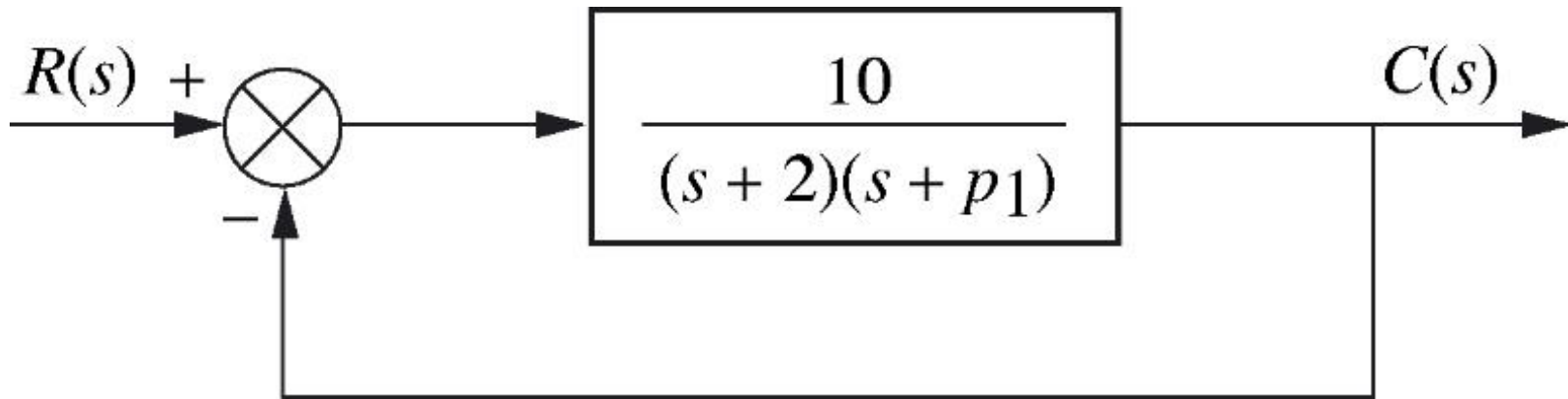
- Substitute  $s=j\omega$  into the characteristic equation and solve for  $\omega$ .

$$\omega = 0 \text{ or } \omega = \pm\sqrt{2}$$



# Generalized Root Locus

Select  $p_1$  to have closed loop damping ration **0.707**.



$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

$$= \frac{10}{s^2 + 2s + 10 + p_1(s + 2)}$$

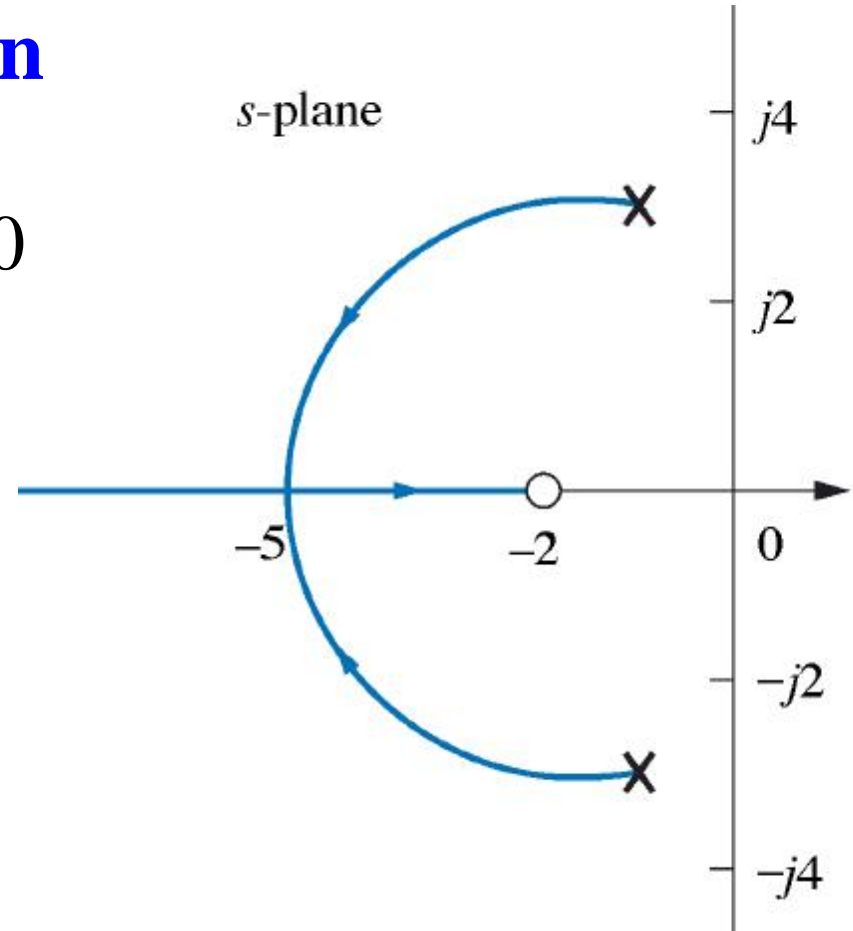
# Generalized Root Locus

## Characteristic equation

$$s^2 + 2s + 10 + p_1(s + 2) = 0$$

$$1 + p_1 \frac{(s + 2)}{s^2 + 2s + 10} = 0$$

$$\underbrace{1 + p_1}_{K} \underbrace{\frac{(s + 2)}{s^2 + 2s + 10}}_{GH} = 0$$



# Generalized Root Locus

**Characteristic equation**  $s^2 + 2s + 10 + p_1(s + 2) = 0$

➔  $s^2 + (2 + p_1)s + (10 + 2p_1) = 0$

**or**  $(10 + 2p_1) = \omega_n^2$ , and  $(2 + p_1) = 2\zeta\omega_n$

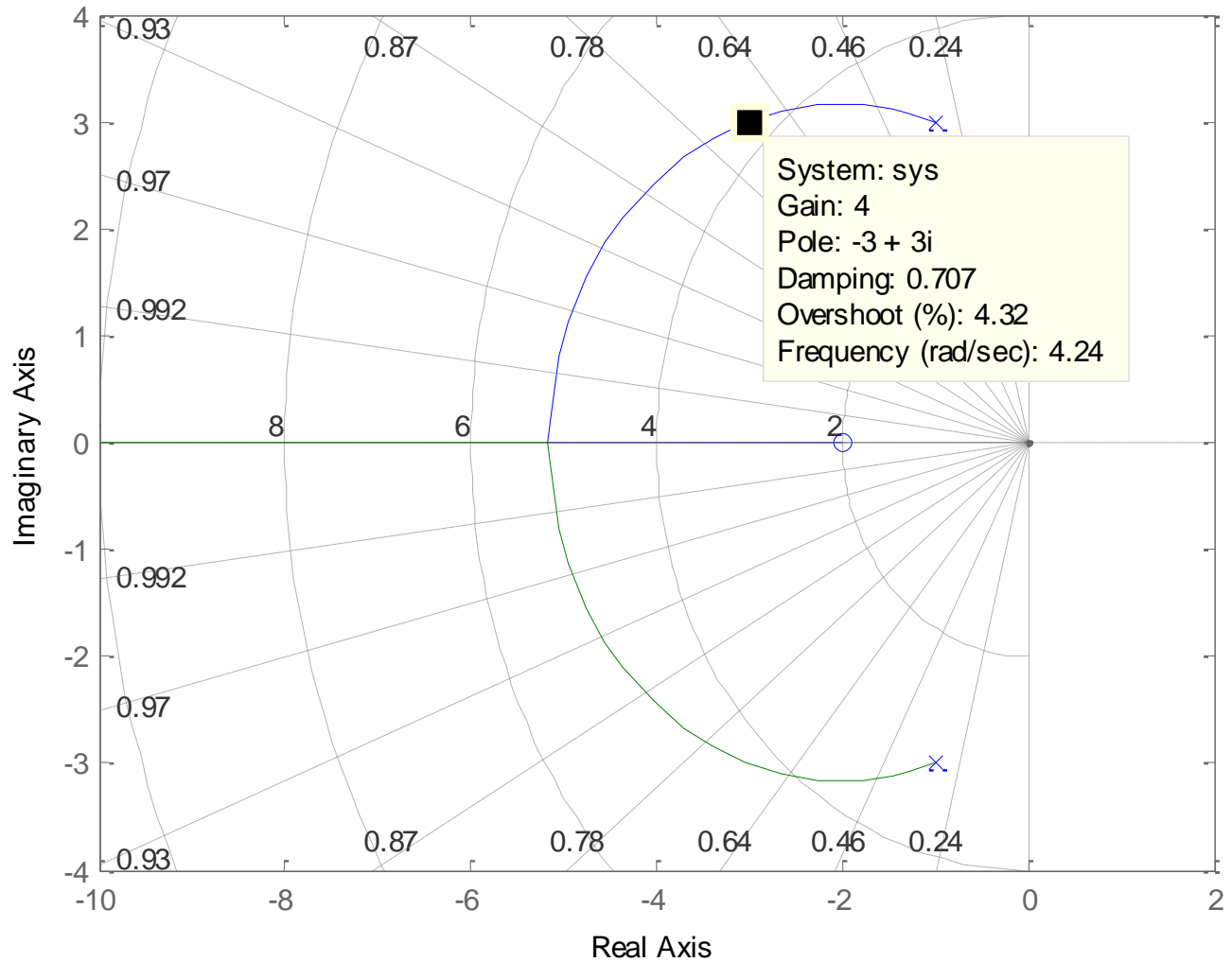
➔  $4\zeta^2(10 + 2p_1) = (2 + p_1)^2$

**or**  $20 + 4p_1 = 4 + 4p_1 + p_1^2$

➔  $p_1 = 4$

# Generalized Root Locus

```
>> n=[1 2];
>> d=[1 2 10];
>> rlocus(n,d)
>> grid
```





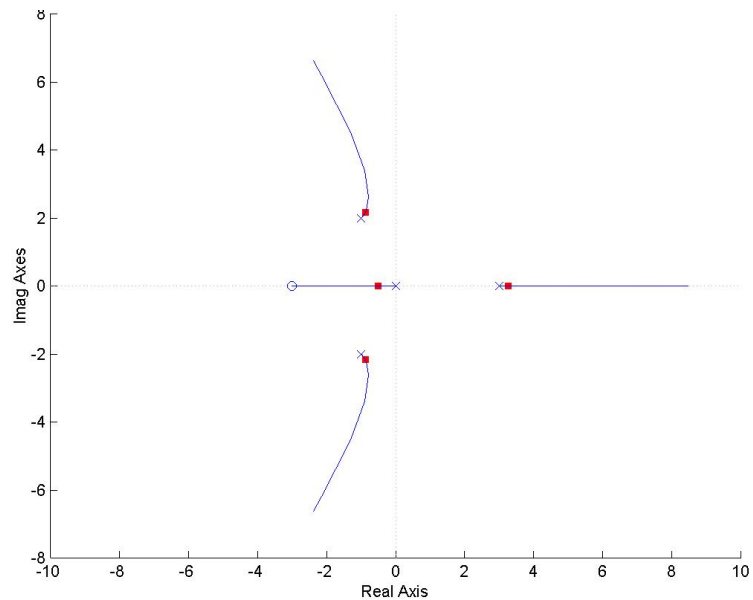
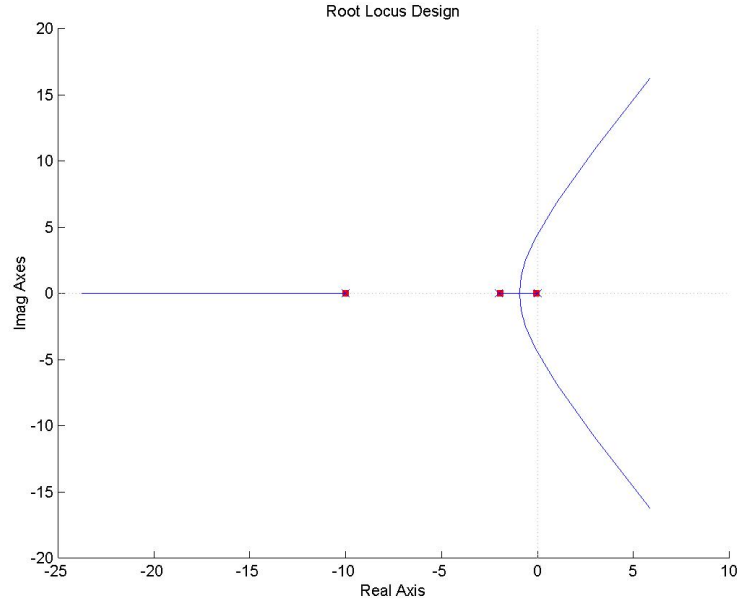
# MATLAB method

$$kGH(s) = \frac{k}{s(1 + 0.5s)(1 + 0.1s)}$$

```
gh=zpk([], [0 -2 -10], [1])
rtool(gh)
```

$$kGH(s) = \frac{k(-3s - 9)}{s^4 - s^3 - s^2 - 15s}$$

```
n=[-3 -9]
m=[1 -1 -1 -15 0]
gh=tf(n,m)
rtool(gh)
```



**TABLE 7-1 Properties of the Root Loci of  $1 + KG_1(s)H_1(s) = 0$**

1. $K = 0$ points	The $K = 0$ points are at the poles of $G(s)H(s)$ , including those at $s = \infty$ .
2. $K = \pm\infty$ points	The $K = \infty$ points are at the zeros of $G(s)H(s)$ , including those at $s = \infty$ .
3. Number of separate root loci	The total number of root loci is equal to the order of the equation $1 + KG_1(s)H_1(s) = 0$ .
4. Symmetry of root loci	The root loci are symmetrical about the axes of symmetry the of pole-zero configuration of $G(s)H(s)$ .
5. Asymptotes of root loci as $s \rightarrow \infty$	For large values of $s$ , the RL ( $K > 0$ ) are asymptotic to asymptotes with angles given by $\theta_i = \frac{2i+1}{ n-m } \times 180^\circ$ For $K < 0$ , the RL are asymptotic to $\theta_i = \frac{2i}{ n-m } \times 180^\circ$ where $i = 0, 1, 2, \dots,  n - m  - 1$ , $n = \text{number of finite poles of } G(s)H(s), \text{ and}$ $m = \text{number of finite zeros of } G(s)H(s).$
6. Intersection of the asymptotes	(a) The intersection of the asymptotes lies only on the real axis in the $s$ -plane. (b) The point of intersection of the asymptotes is given by $\sigma_1 = \frac{\sum \text{real parts of poles of } G(s)H(s) - \sum \text{real parts of zeros of } G(s)H(s)}{n-m}$
7. Root loci on the real axis.	RL for $K \geq 0$ are found in a section of the real axis only if the total number of real poles and zeros of $G(s)H(s)$ to the <b>right</b> of the section is <b>odd</b> . If the total number of real poles and zeros to the right of a given section is <b>even</b> , RL for $K \leq 0$ are found.

(Continued)

**TABLE 7-1 (Continued)**

8. Angles of departure

The angle of departure or arrival of the RL from a pole or a zero of  $G(s)H(s)$  can be determined by assuming a point  $s_1$  that is very close to the pole, or zero, and applying the equation

$$\begin{aligned} \angle G(s_1)H(s_1) &= \sum_{k=1}^m \angle(s_1 - z_k) - \sum_{j=1}^m \angle(s_1 - p_j) \\ &= 2(i + 1)180^\circ \quad K \geq 0 \\ &= 2i \times 180^\circ \quad K \leq 0 \end{aligned}$$

where  $i = 0, \pm 1, \pm 2, \dots$

9. Intersection of the root loci with the imaginary axis

The crossing points of the root loci on the imaginary axis and the corresponding values of  $K$  may be found by use of the Routh-Hurwitz criterion.

10. Breakaway points

The breakaway points on the root loci are determined by finding the roots of  $dK/ds = 0$ , or  $dG(s)H(s)/ds = 0$ . These are necessary conditions only.

11. Calculation of the values of  $K$

The absolute value of  $K$  at any point  $s_1$  on the root loci is on the root loci determined from the equation

$$|K| = \frac{1}{|G_1(s_1)H_1(s_1)|}$$

table\_07\_01b

Draw root locus for the following CE of a closed loop TF:

$$s(s + 5)(s + 6)(s^2 + 2s + 2) + K(s + 3) = 0$$

► **EXAMPLE E-10-2** Consider the equation

$$s(s+5)(s+6)(s^2+2s+2) + K(s+3) = 0 \quad (\text{E-42})$$

Dividing both sides of the last equation by the terms that do not contain  $K$ , we have

$$G(s)H(s) = \frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)} \quad (\text{E-43})$$

The following properties of the root loci are determined:

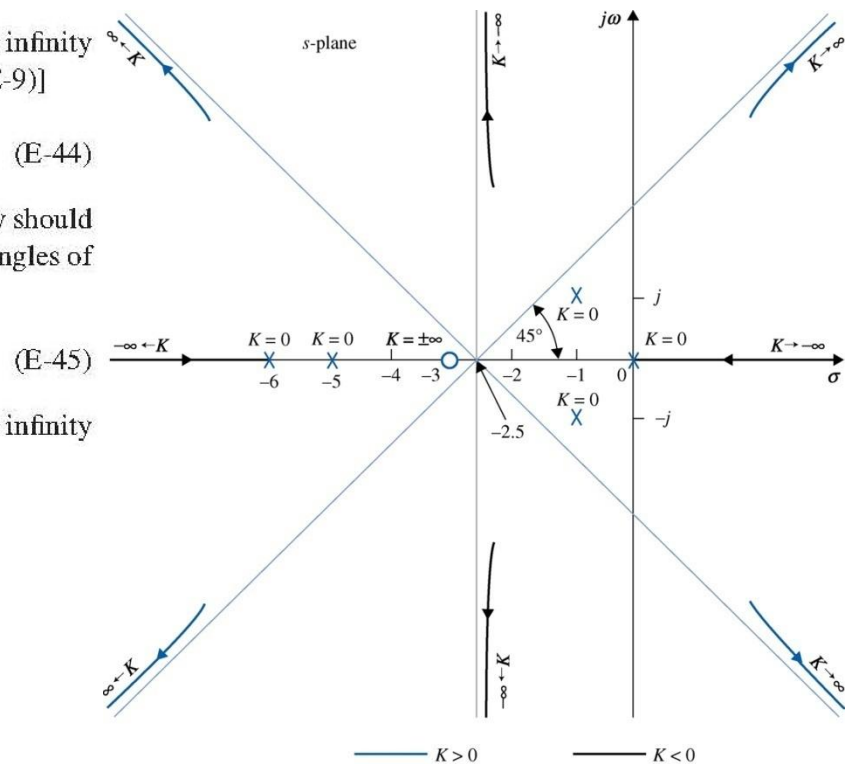
1. The  $K = 0$  points are at the poles of  $G(s)H(s)$ :  $s = -5, -6, -1 + j$ , and  $-1 - j$ .
2. The  $K = \pm \infty$  points are at the zeros of  $G(s)H(s)$ :  $s = -3, \infty, \infty, \infty, \infty$ .
3. There are five separate branches on the root loci.
4. The root loci are symmetrical with respect to the real axis of the  $s$ -plane.
5. Since  $G(s)H(s)$  has five poles and one finite zero, four RL and CRL should approach infinity along the asymptotes. The angles of the asymptotes of the RL are given by [Eq. (E-9)]

$$\theta_i = \frac{2i+1}{|n-m|} 180^\circ = \frac{2i+1}{|5-1|} 180^\circ \quad 0 \leq K < \infty \quad (\text{E-44})$$

for  $i = 0, 1, 2, 3$ . Thus, the four root loci that approach infinity as  $K$  approaches infinity should approach asymptotes with angles of  $45^\circ, -45^\circ, 135^\circ$ , and  $-135^\circ$ , respectively. The angles of the asymptotes of the CRL at infinity are given by Eq. (E-10):

$$\theta_i = \frac{2i}{|n-m|} 180^\circ = \frac{2i}{|5-1|} 180^\circ \quad -\infty < K \leq 0 \quad (\text{E-45})$$

for  $i = 0, 1, 2, 3$ . Thus, as  $K$  approaches  $-\infty$ , four root loci for  $K < 0$  should approach infinity along asymptotes with angles of  $0^\circ, 90^\circ, 180^\circ$ , and  $270^\circ$ .



6. The intersection of the asymptotes is given by [Eq. (E-12)]

$$\sigma_1 = \frac{\Sigma(-5 - 6 - 1 - 1) - (-3)}{4} = -2.5 \quad (\text{E-46})$$

The results from these six steps are illustrated in Fig. E-15. It should be pointed out that in general the properties of the asymptotes do not indicate on which side of the asymptotes the root loci lie. The asymptotes indicate nothing more than the behavior of the root loci as  $|s| \rightarrow \infty$ . In fact, the root locus can even cross an asymptote in the finite  $s$  domain. The segments of the root loci shown in Fig. E-15 can be accurately plotted only if additional information is obtained.

7. Root loci on the real axis: There are  $K \geq 0$  root loci on the real axis between  $s = 0$  and  $-3$ , and  $s = -5$  and  $-6$ . There are  $K \leq 0$  root loci on the remaining portions of the real axis, that is, between  $s = -3$  and  $-5$ , and  $s = -6$  and  $-\infty$ , as shown in Fig. E-16.

8. Angles of departure: The angle of departure  $\theta$  of the root loci leaving the pole at  $-1 + j$  is determined using Eq. (7-18). If  $s_1$  is a point on the root loci leaving the pole at  $-1 + j$ , and  $s_1$  is very close to the pole, as shown in Fig E-17, Eq. (7-18) gives

$$\angle(s_1 + 3) - \angle s_1 - \angle(s_1 + 1 + j) - \angle(s_1 + 5) - \angle(s_1 + 1 - j) = (2i + 1)180^\circ \quad (\text{E-47})$$

or

$$26.6^\circ - 135^\circ - 90^\circ - 14^\circ - 11.4^\circ - \theta \cong (2i + 1)180^\circ$$

for  $i = 0, \pm 1, \pm 2, \dots$ . Therefore, selecting  $i = 2$ ,

$$\theta \cong -43.8^\circ$$

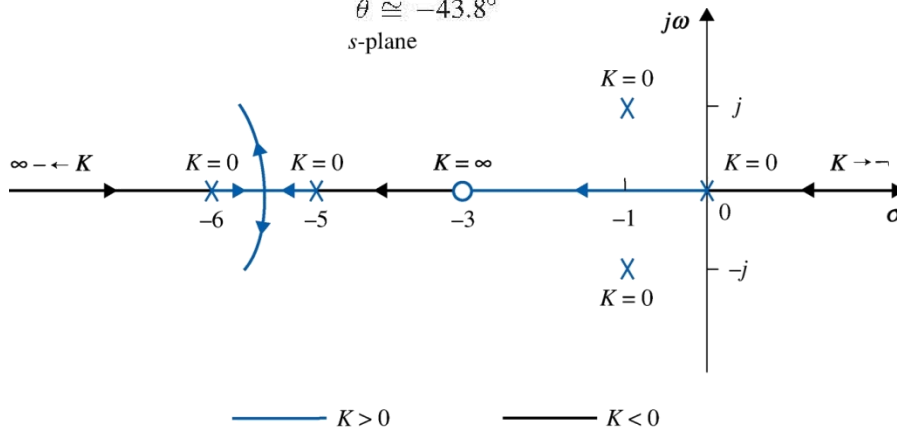


Figure E-16 Root loci of  $s(s + 5)(s + 6)(s^2 + 2s + 2) + K(s + 3) = 0$  on the real axis.

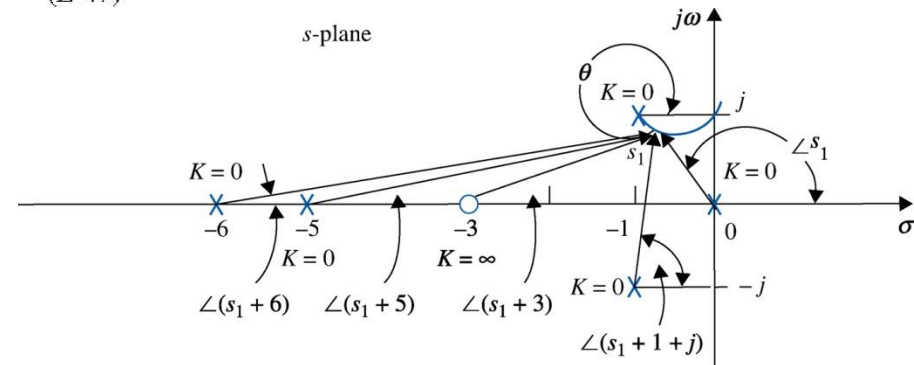


Figure E-17 Computation of angle of departure of the root loci of  $s(s + 5)(s + 6)(s^2 + 2s + 2) + K(s + 3) = 0$ .



Similarly, Eq. (7-19) is used to determine the angle of arrival  $\theta'$  of the  $K \leq 0$  root loci arriving at the pole  $-1 + j$ . It is easy to see that  $\theta'$  differs from  $\theta$  by  $180^\circ$ ; thus,

$$\theta' = 180^\circ - 43.8^\circ = 136.2^\circ \quad (\text{E-50})$$

9. The intersection of the root loci on the imaginary axis is determined using Routh's tabulation. Equation (E-42) is written

$$s^5 + 13s^4 + 54s^3 + 82s^2 + (60 + K)s + 3K = 0 \quad (\text{E-51})$$

Routh's tabulation is

$s^5$	1	54	$60 + K$
$s^4$	13	82	$3K$
$s^3$	47.7	$0.769K$	0
$s^2$	$65.6 - 0.212K$	$3K$	0
$s^1$	$\frac{3940 - 105K - 0.163K^2}{65.6 - 0.212K}$	0	0
$s^0$	$3K$	0	0

For Eq. (E-51) to have no roots on the imaginary axis or in the right-half of the  $s$ -plane, the elements in the first column of Routh's tabulation must all be of the same sign. Thus, the following inequalities must be satisfied:

$$65.6 - 0.212K > 0 \quad \text{or} \quad K < 309 \quad (\text{E-52})$$

$$3940 - 105K - 0.163K^2 > 0 \quad \text{or} \quad K < 35 \quad (\text{E-53})$$

$$K > 0 \quad (\text{E-54})$$

Thus, all the roots of Eq. (E-51) will stay in the left-half  $s$ -plane if  $K$  lies between 0 and 35, which means that the root loci of Eq. (E-51) cross the imaginary axis when  $K = 35$  and  $K = 0$ . The coordinates at the crossover points on the imaginary axis that correspond to  $K = 35$  are determined from the auxiliary equation:

$$A(s) = (65.6 - 0.212K)s^2 + 3K = 0 \quad (\text{E-55})$$

which is obtained by using the coefficients from the row just above the row of zeros in the  $s^1$  row that would have happened when  $K$  is set to 35. Substituting  $K = 35$  in Eq. (E-55), we get

$$58.2s^2 + 105 = 0 \quad (\text{E-56})$$

The roots of Eq. (E-56) are  $s = j1.34$  and  $-j1.34$ , which are the points at which the root loci cross the  $j\omega$ -axis.

0. Breakaway points: Based on the information gathered from the preceding nine steps, a trial sketch of the root loci indicates that there can be only one breakaway point on the entire root loci, and the point should lie between the two poles of  $G(s)H(s)$  at  $s = -5$  and  $-6$ . To find the breakaway point, we take the derivative on both sides of Eq. (E-43) with respect to  $s$  and set it to zero; the resulting equation is

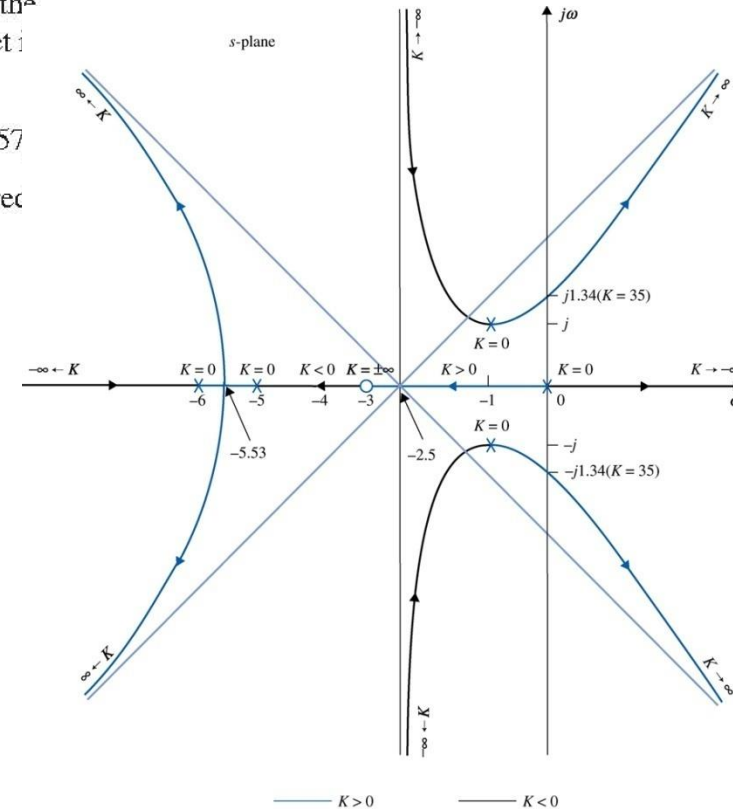
$$s^5 + 13.5s^4 + 66s^3 + 142s^2 + 123s + 45 = 0 \quad (\text{E-57})$$

Since there is only one breakaway expected, only one root of the last equation is the correct solution of the breakaway point. The five roots of Eq. (E-57) are:

$$\begin{aligned} s &= 3.33 + j1.204 & s &= 3.33 - j1.204 \\ s &= -0.656 + j0.468 & s &= -0.656 - j0.468 \\ s &= -5.53 \end{aligned}$$

**Figure E-18** Root loci of  $s(s + 5)(s + 6)(s^2 + 2s + 2) + K(s + 3) = 0$ .

Clearly, the breakaway point is at  $-5.53$ . The other four solutions do not satisfy Eq. (E-51) and are not breakaway points. Based on the information obtained in the last 10 steps, the root loci of Eq. (E-51) are sketched as shown in Fig. E-18





# Root Locus Sketching Rules

$$1 + K \cdot \frac{N(s)}{D(s)} = 0 \quad \Rightarrow \quad 1 + K \cdot \frac{(s - z_1)(s - z_2) \cdots (s - z_{N_z})}{(s - p_1)(s - p_2) \cdots (s - p_{N_p})} = 0$$

**Rule 1:** The number of branches of the root locus is equal to the number of closed-loop poles (or roots of the characteristic equation). In other words, the number of branches is equal to the number of open-loop poles or open-loop zeros, whichever is greater.  $D(s) + KN(s) = 0$

**Rule 2:** Root locus starts at open-loop poles (*when  $K=0$* ) and ends at open-loop zeros (*when  $K=\infty$* ). If the number of open-loop poles is greater than the number of open-loop zeros, some branches starting from finite open-loop poles will terminate at zeros at infinity (i.e., go to infinity). If the reverse is true, some branches will start at poles at infinity and terminate at the finite open-loop zeros.

$$D(s) + KN(s) = 0 \quad \boxed{K = 0?} \quad \boxed{K = \infty?}$$

**Rule 3:** Root locus is symmetric about the real axis, which reflects the fact that closed-loop poles appear in complex conjugate pairs.

**Rule 4:** Along the real axis, the root locus includes all segments that are to the left of an *odd* number of finite real open-loop poles and zeros.

Check the phases

$$\angle K \frac{N(s)}{D(s)} = \angle -1 = \pi [\text{rad}] = 180^\circ$$

# Root Locus Sketching Rules

Rule 5: If number of poles  $N_p$  exceeds the number of zeros  $N_z$ , then as  $K \rightarrow \infty$ ,  $(N_p - N_z)$  branches will become asymptotic to straight lines. These straight lines intersect the real axis with angles  $\theta_k$  at  $\sigma_0$ .

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_p - N_z} = \frac{\text{Sum of open-loop poles} - \text{Sum of open-loop zeros}}{\# \text{ of open-loop poles} - \# \text{ of open-loop zeros}}$$

$$\theta_k = (2k + 1) \frac{\pi}{N_p - N_z} \text{ [rad]} = (2k + 1) \frac{180^\circ}{N_p - N_z} \text{ [deg]}, \quad k = 0, 1, 2, \dots$$

If  $N_z$  exceeds  $N_p$ , then as  $K \rightarrow 0$ ,  $(N_z - N_p)$  branches behave as above.

Rule 6: Breakaway and/or break-in (arrival) points should be the solutions to the following equations:

$$\frac{d}{ds} \left( \frac{N(s)}{D(s)} \right) = 0 \quad \text{or} \quad \frac{d}{ds} \left( \frac{D(s)}{N(s)} \right) = 0$$

# Root Locus Sketching Rules



Rule 7: The departure angle for a pole  $p_i$  ( the arrival angle for a zero  $z_j$ ) can be calculated by slightly modifying the following equation:

*angle criterion* → 
$$\angle(s - z_1) + \angle(s - z_2) + \dots + \angle(s - z_{N_z}) - \angle(s - p_1) - \angle(s - p_2) - \dots - \angle(s - p_{N_p}) = 180^\circ$$

The departure angle  $q_j$  from the pole  $p_j$  can be calculated by replacing the term  $\angle(s - p_j)$  with  $q_j$  and replacing all the  $s$ 's with  $p_j$  in the other terms.

Rule 8: If the root locus passes through the imaginary axis (the stability boundary), the crossing point  $j\omega$  and the corresponding gain  $K$  can be found as follows:

- Replace  $s$  in the left side of the closed-loop characteristic equation with  $j\omega$  to obtain the real and imaginary parts of the resulting complex number.
- Set the real and imaginary parts to zero, and solve for  $\omega$  and  $K$ . This will tell you at what values of  $K$  and at what points on the  $j\omega$  axis the roots will cross.

*magnitude criterion* → 
$$K = \frac{|s - p_1| |s - p_2| \cdots |s - p_{N_p}|}{|s - z_1| |s - z_2| \cdots |s - z_{N_z}|}$$

# Steps to Sketch Root Locus



Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1 + K \cdot \frac{N(s)}{D(s)} = 0 \quad \text{or} \quad 1 + K \cdot \frac{(s - z_1)(s - z_2) \cdots (s - z_{N_z})}{(s - p_1)(s - p_2) \cdots (s - p_{N_p})} = 0$$

Step 2: Find the open-loop zeros,  $z_j$ , and the open-loop poles,  $p_j$ . Mark the open-loop poles and zeros on the complex plane. Use  $\times$  to represent open-loop poles and  $o$  to represent the open-loop zeros.

Step 3: Determine the real axis segments that are on the root locus by applying Rule 4.

Step 4: Determine the number of asymptotes and the corresponding intersection  $\sigma_0$  and angles  $\theta_k$  by applying Rules 2 and 5.

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

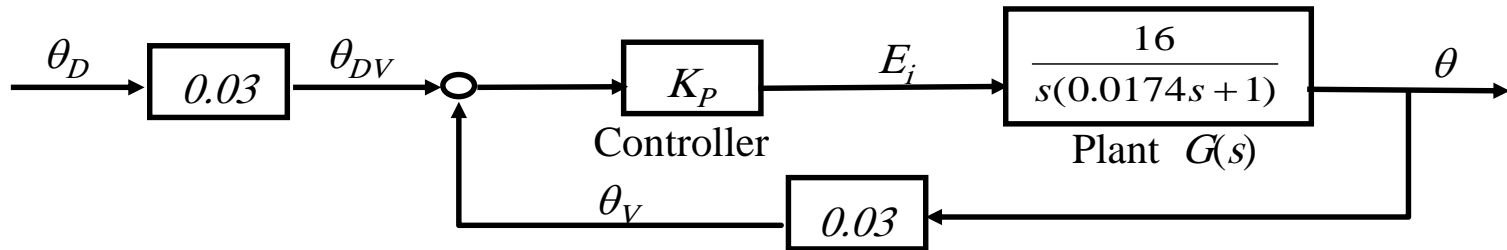
Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.

# Example 1

## DC Motor Position Control

In the printer paper advance position control, the proportional control block diagram is:



Sketch the root locus of the closed-loop poles as the proportional gain  $K_p$  varies from 0 to  $\infty$ .

*Find closed-loop characteristic equation:*

$$1 + K_p G(s) H(s) = 0$$

$$1 + K_p \frac{0.48 N(s)}{\underbrace{s(0.0174s + 1)}_{D(s)}} = 0$$

# Example 1

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1 + \frac{27.58K_p}{K} \frac{1}{\underbrace{s(s + 57.47)}_{D(s)}} = 0$$

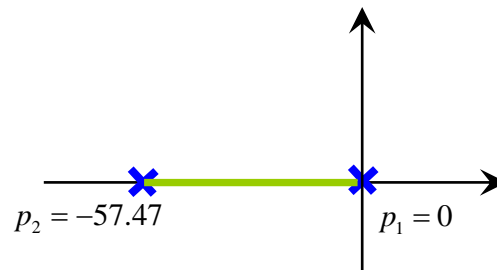
Step 2: Find the open-loop zeros,  $z_i$ , and the open-loop poles,  $p_i$ :

No open-loop zeros

open-loop poles

$$p_1 = 0, p_2 = -57.47$$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



# Example 1

*Step 4: Determine the number of asymptotes and the corresponding intersection  $\sigma_0$  and angles  $\theta_k$  by applying Rules 2 and 5.*

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_p - N_z} = \frac{-57.47}{2} = -28.74$$

$$\theta_k = (2k + 1) \frac{\pi}{N_p - N_z} \text{ [rad]} = \begin{cases} \frac{\pi}{2} \\ \frac{3\pi}{2} \end{cases}$$

*Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.*

$$\frac{d}{ds} \left( \frac{N(s)}{D(s)} \right) = 0 \text{ or } \frac{d}{ds} \left( \frac{D(s)}{N(s)} \right) = 0,$$

$$\frac{d}{ds} \left( \frac{s(0.0174s + 1)}{0.48} \right) = 0, 0.0348s + 1 = 0, s = -28.74$$

*Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.*

$$-\angle(p_2 - p_1) - \theta_{p_2} = 180^\circ, \theta_{p_2} = 0^\circ$$

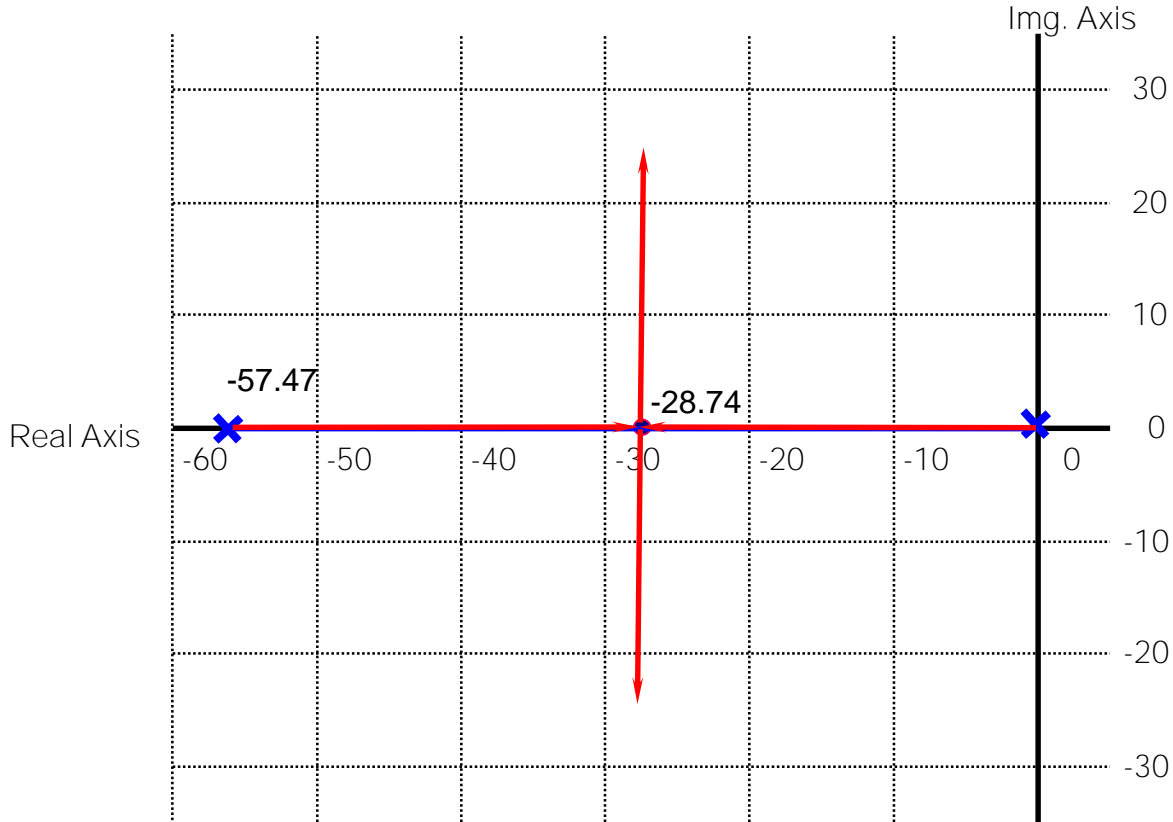
$$-\theta_{p_1} - \angle(p_1 - p_2) = 180^\circ, \theta_{p_1} = 180^\circ$$

*Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.*

*Could  $s$  be pure imaginary in this example?*

# Example 1

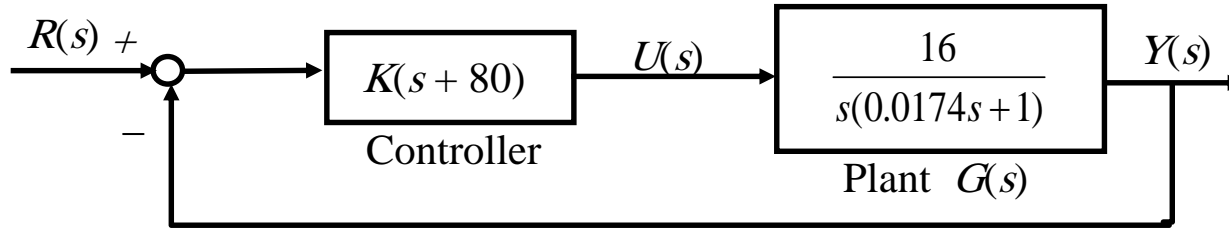
Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.





# Example 2

A positioning feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain  $K$  varies from 0 to  $\infty$ .

*Find closed-loop characteristic equation:*

$$1 + G_c(s)G(s)H(s) = 0$$

$$1 + K(s+80) \frac{16}{s(0.0174s+1)} = 0$$

# Example 2

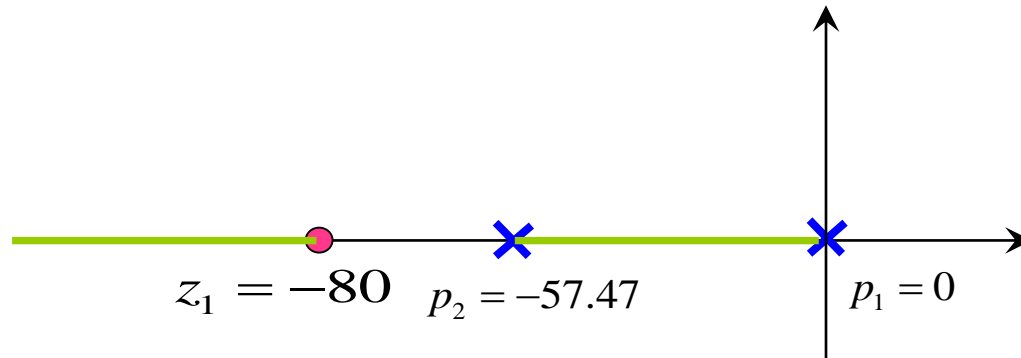
Step 1: Formulate the (closed-loop) characteristic equation into the standard form for sketching root locus:

$$1 + K \frac{\underbrace{16(s + 80)}_{N(s)}}{\underbrace{s(0.0174s + 1)}_{D(s)}} = 1 + \boxed{920K} \frac{\underbrace{(s + 80)}_{N(s)}}{\underbrace{s(s + 57.47)}_{D(s)}} = 0$$

Step 2: Find the open-loop zeros,  $z_i$ , and the open-loop poles,  $p_i$ :

open-loop zeros	$z_1 = -80$
open-loop poles	$p_1 = 0, p_2 = -57.47$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



# Example 2

Step 4: Determine the number of asymptotes and the corresponding intersection  $\sigma_0$  and angles  $\theta_k$  by applying Rules 2 and 5.

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds} \left( \frac{N(s)}{D(s)} \right) = 0 \quad \text{or} \quad \frac{d}{ds} \left( \frac{D(s)}{N(s)} \right) = 0,$$

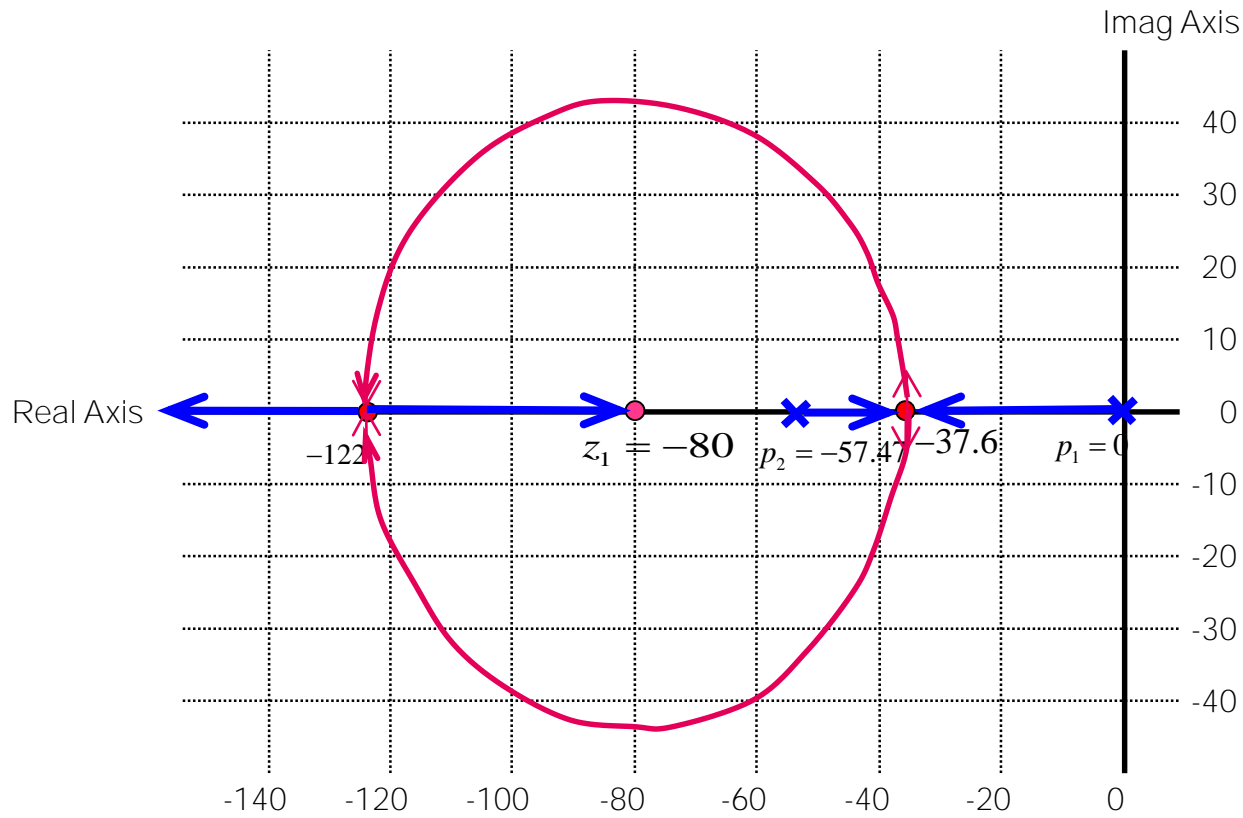
$$\frac{d}{ds} \left( \frac{(s+80)}{s(s+57.47)} \right) = \frac{s(s+57.47) - (s+80)(2s+57.47)}{[s(s+57.47)]^2} = 0,$$

$$s^2 + 160s + 4600 = 0$$

$$s_1 = -122, s_2 = -37.6$$

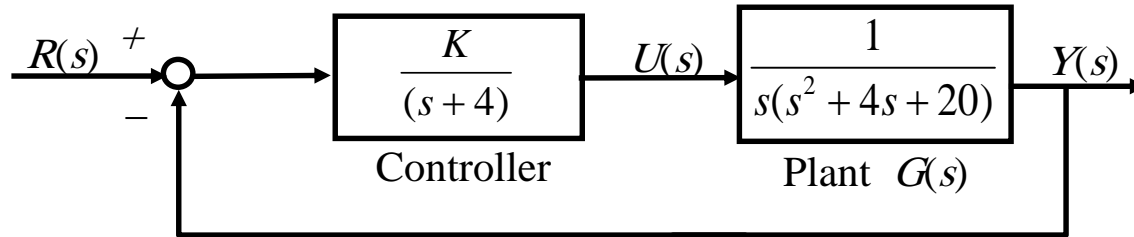
# Example 2

- Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.
- Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.
- Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.



# Example 3

A feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain  $K$  varies from 0 to  $\infty$ .

*Find closed-loop characteristic equation:*

$$1 + G_c(s)G(s)H(s) = 0$$

$$1 + \frac{K}{s+4} \frac{1}{s(s^2+4s+20)} = 0$$

# Example 3

*Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:*

$$1 + K \frac{1}{\underbrace{s(s^2 + 4s + 20)}_{D(s)}} = 0$$

*Step 2: Find the open-loop zeros,  $z_i$ , and the open-loop poles,  $p_i$ :*

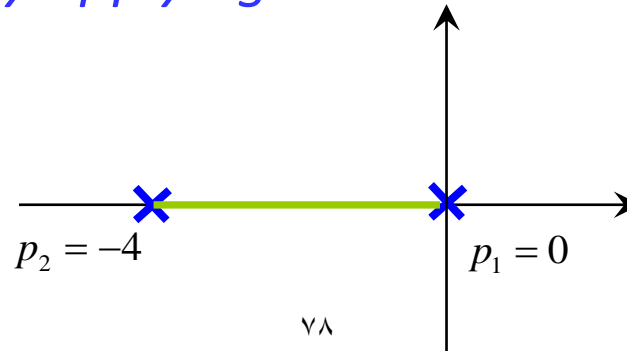
open-loop zeros

No open-loop zeros

open-loop poles

$p_1 = 0, p_2 = -4, p_{3,4} = -2 \pm 4j$

*Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.*



# Example 3



Step 4: Determine the number of asymptotes and the corresponding intersection  $\sigma_0$  and angles  $\theta_k$  by applying Rules 2 and 5.

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = \frac{0 + (-4) + (-2 + 4j) + (-2 - 4j)}{4 - 0} = -2$$

$$\theta_k = (2k + 1) \frac{\pi}{N_P - N_Z} \text{ [rad]} = \begin{cases} \frac{\pi}{4} \\ \frac{3\pi}{4} \\ \frac{5\pi}{4} \\ \frac{7\pi}{4} \end{cases}$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds} \left( \frac{N(s)}{D(s)} \right) = 0 \quad \text{or} \quad \frac{d}{ds} \left( \frac{D(s)}{N(s)} \right) = 0,$$

$$\frac{d}{ds} \left( \frac{D(s)}{N(s)} \right) = \frac{d}{ds} \left( \frac{s(s^2 + 4s + 20)(s + 4)}{1} \right) = \frac{d}{ds} (s^4 + 8s^3 + 36s^2 + 80s)$$

$$= 4s^3 + 24s^2 + 72s + 80 = 0$$

$$s_1 = -2, s_{2,3} = -2 \pm 2.45j$$

# Example 3

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

$$\sum_{i=1}^{N_z} \angle(s - z_i) - \sum_{i=1}^{N_p} \angle(s - p_i) = 180^\circ$$

$$p_1 = 0: \quad \theta_{p_1} = 180^\circ \quad p_3 = -2 + 4j: \quad \theta_{p_3} = -90^\circ$$

$$p_2 = -4: \quad \theta_{p_2} = 0^\circ \quad p_4 = -2 - 4j: \quad \theta_{p_4} = 90^\circ$$

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

$$1 + K \frac{1}{s(s^2 + 4s + 20)(s + 4)} = 0 \Rightarrow$$

$$s(s^2 + 4s + 20)(s + 4) + K = 0$$

$$\Leftrightarrow s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

CLCE

$$s = j\omega$$

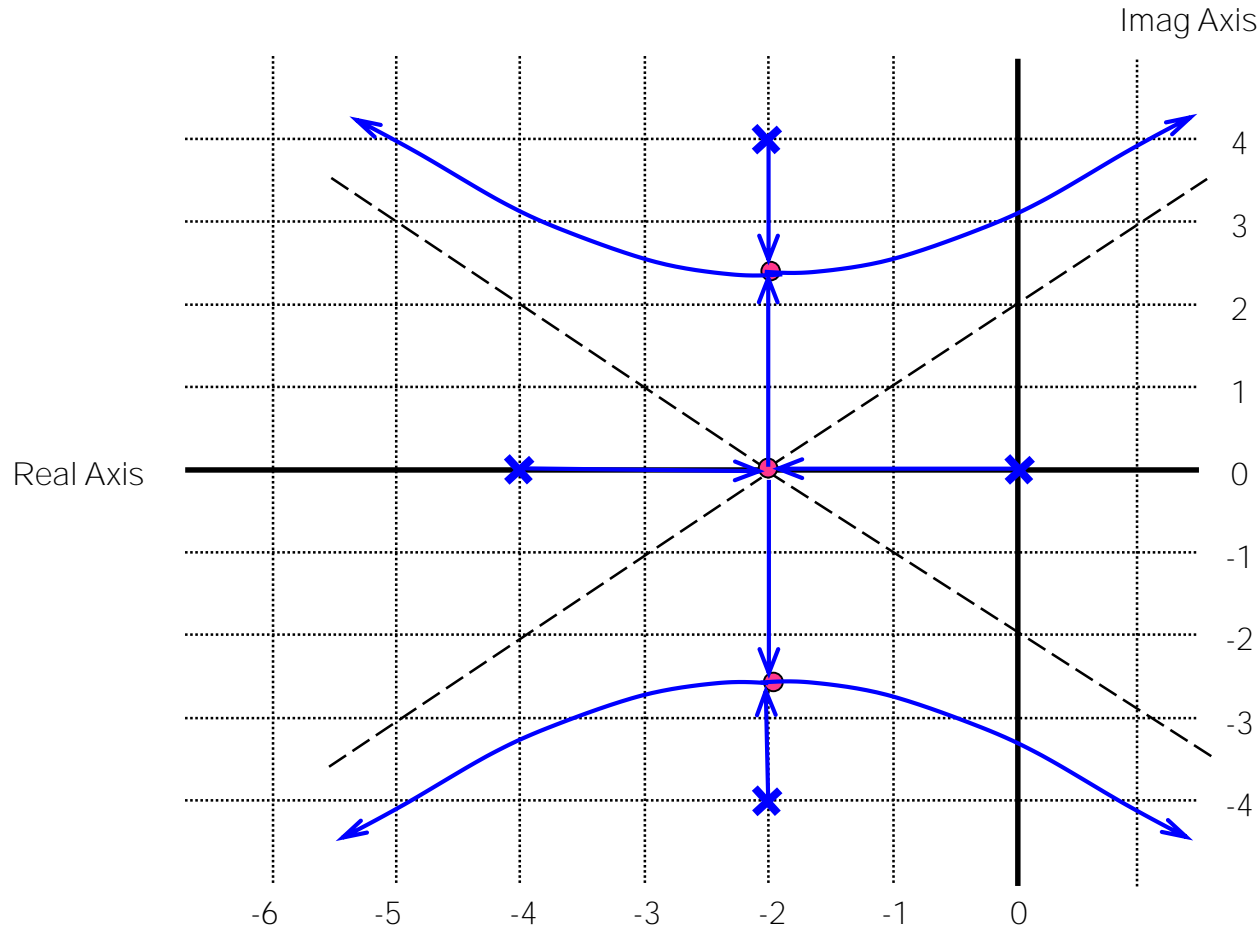
$$\Rightarrow (\omega^4 - 36\omega^2 + K) + (-8\omega^3 + 80\omega)j = 0$$

$$\Rightarrow \begin{cases} \omega^4 - 36\omega^2 + K = 0 \\ -8\omega^3 + 80\omega = 0 \end{cases} \Rightarrow \begin{cases} K_1 = 0 \\ \omega_1 = 0 \end{cases}, \begin{cases} K_2 = 260 \\ \omega_2 = \sqrt{10} = 3.16 \end{cases}$$



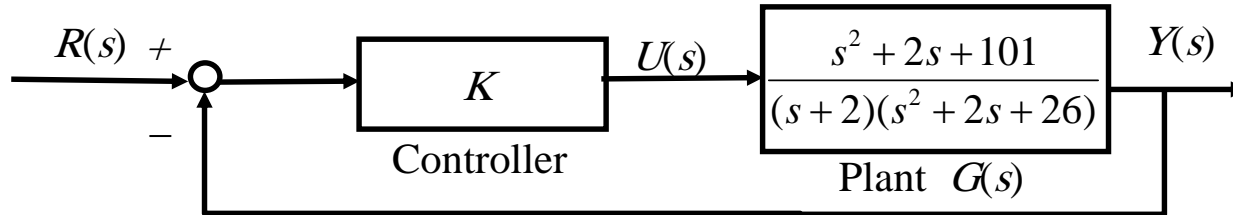
# Example 3

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.



# Example 4

A feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain  $K$  varies from 0 to  $\infty$ .

*Find closed-loop characteristic equation:*

$$1 + K \frac{s^2 + 2s + 101}{(s + 2)(s^2 + 2s + 26)} = 0$$

# Example 4

Step 1: Formulate the (closed-loop) characteristic equation into the standard form for sketching root locus:

$$1 + K \frac{\overbrace{s^2 + 2s + 101}^{N(s)}}{\underbrace{(s + 2)(s^2 + 2s + 26)}_{D(s)}} = 0$$

Step 2: Find the open-loop zeros,  $z_i$ , and the open-loop poles,  $p_i$ :

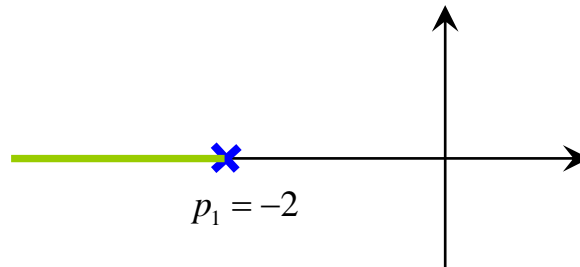
open-loop zeros

$$s^2 + 2s + 101 = (s + 1)^2 + 100 = 0, z_{1,2} = -1 \pm 10j$$

open-loop poles

$$(s + 2)((s + 1)^2 + 25) = 0, p_1 = -2, p_{2,3} = -1 \pm 5j$$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



# Example 4

Step 4: Determine the number of asymptotes and the corresponding intersection  $\sigma_0$  and angles  $\theta_k$  by applying Rules 2 and 5.

$$N_p - N_z = 1 \quad \text{One asymptote} \quad \theta_k = (2k + 1) \times 180^\circ = 180^\circ$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

$$z_1 = -1 + 10j \quad \theta_{z_1} = 90^\circ - \tan^{-1}(10) - 90^\circ - 90^\circ = 180^\circ \quad p_1 = -2 \quad \theta_{p_1} = 180^\circ$$

$$\theta_{z_1} = 354^\circ = -6^\circ \quad p_2 = -1 + 5j \quad \theta_{p_2} = 11^\circ$$

$$z_2 = -1 - 10j \quad \theta_{z_2} = 6^\circ \quad p_3 = -1 - 5j \quad \theta_{p_3} = -11^\circ$$

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

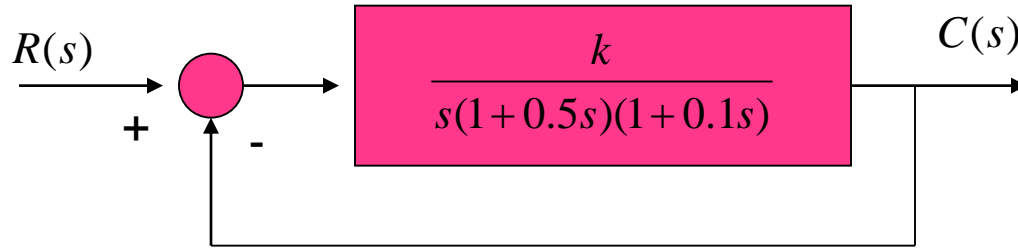
$$(s + 2)(s^2 + 2s + 26) + K(s^2 + 2s + 101) = 0$$

$$\Leftrightarrow s^3 + (4 + K)s^2 + (30 + 2K)s + (52 + 101K) = 0$$

$$\Rightarrow \overset{s=j\omega}{\Rightarrow} [(52 + 101K) - (4 + K)\omega^2] + [(30 + 2K) - \omega^2]\omega j = 0$$

$$\begin{cases} (52 + 101K) - (4 + K)\omega^2 = 0 \\ [(30 + 2K) - \omega^2]\omega = 0 \end{cases} \Rightarrow \begin{cases} \omega_1 = 0 \\ K_1 = -\frac{52}{101} \end{cases}, \begin{cases} \omega_2 = 9.5 \\ K_2 = 30.4 \end{cases}, \begin{cases} \omega_3 = 5.7 \\ K_3 = 1.1 \end{cases}$$

# Example 4



$$kGH(s) = \frac{k}{s(1+0.5s)(1+0.1s)}$$

(i)  $poles = 0, -2, -10$   
 $zeros = \infty, \infty, \infty$

(ii)  $\sigma = \frac{0 + (-2) + (-10) - 0}{3 - 0} = -4$   
 $\theta_k = \frac{180}{3 - 0} = 60$

$$s(1+0.5s)(1+0.1s) + k = 0$$

$$0.05s^3 + 0.6s^2 + s + k = 0$$

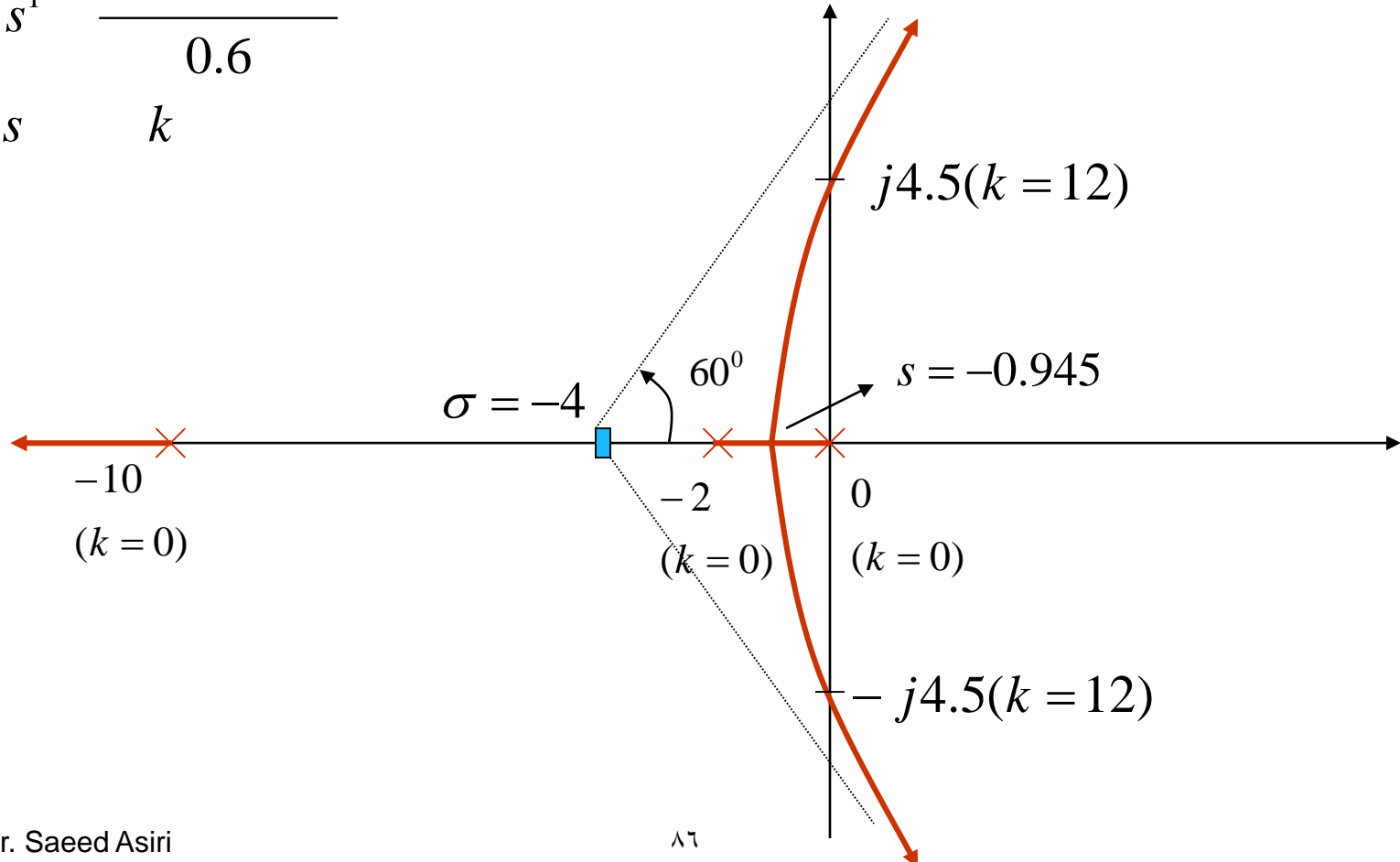
(iii)  $\frac{dk}{ds} = -\frac{d}{ds} (0.05s^3 + 0.6s^2 + s) = 0$   
 $s_1 = -0.945, s_2 = -7.05$

# Example 4

$$0.05s^3 + 0.6s^2 + s + k = 0$$

$$k = 12$$

$s^3$	0.05	1	→	$0.6s^2 + 12 = 0$
$s^2$	0.6	$k$		$s = \pm j4.5$
$s^1$	$\frac{0.6 - 0.05k}{0.6}$			
$s$	$k$			

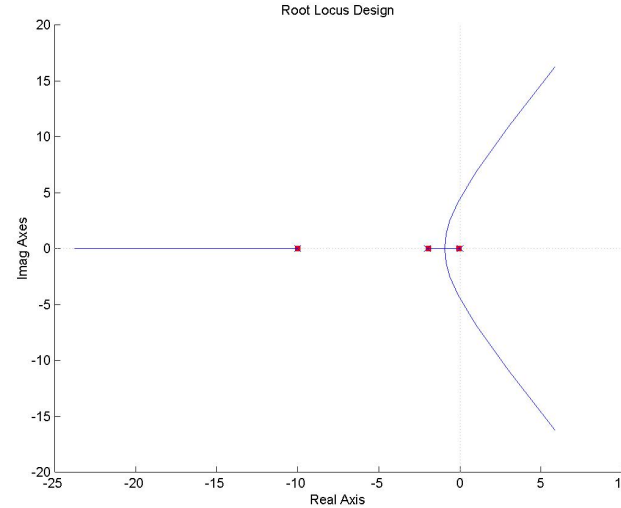


# Example 4

## MATLAB method

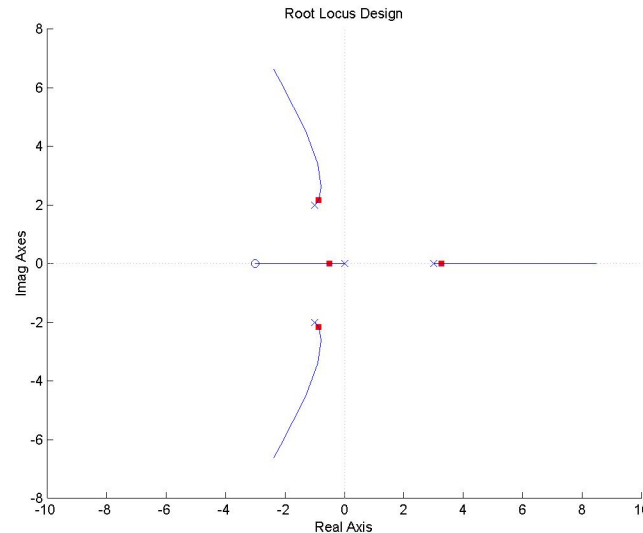
$$kGH(s) = \frac{k}{s(1+0.5s)(1+0.1s)}$$

```
gh=zpk([], [0 -2 -10], [1])
rtool(gh)
```



$$kGH(s) = \frac{k(-3s-9)}{s^4 - s^3 - s^2 - 15s}$$

```
n=[-3 -9]
m=[1 -1 -1 -15 0]
gh=tf(n,m)
rtool(gh)
```



# Example 4

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.

Stability condition

$$0 < K < 1.1$$

or

$$K > 30.4$$

