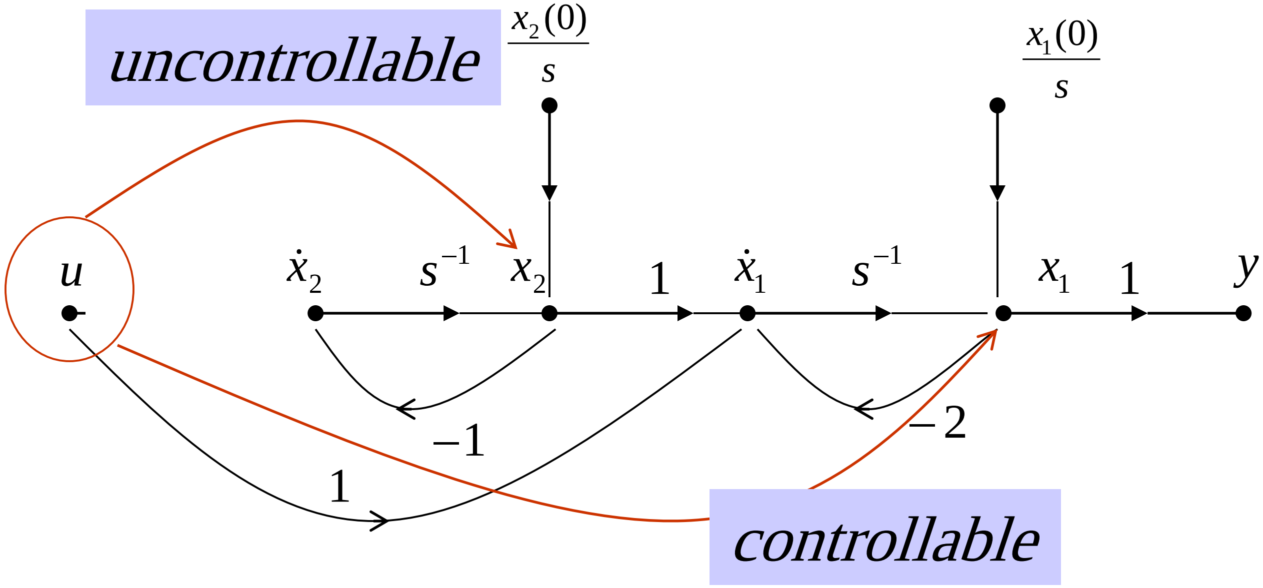
Controllability & Observability

Motivation1

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

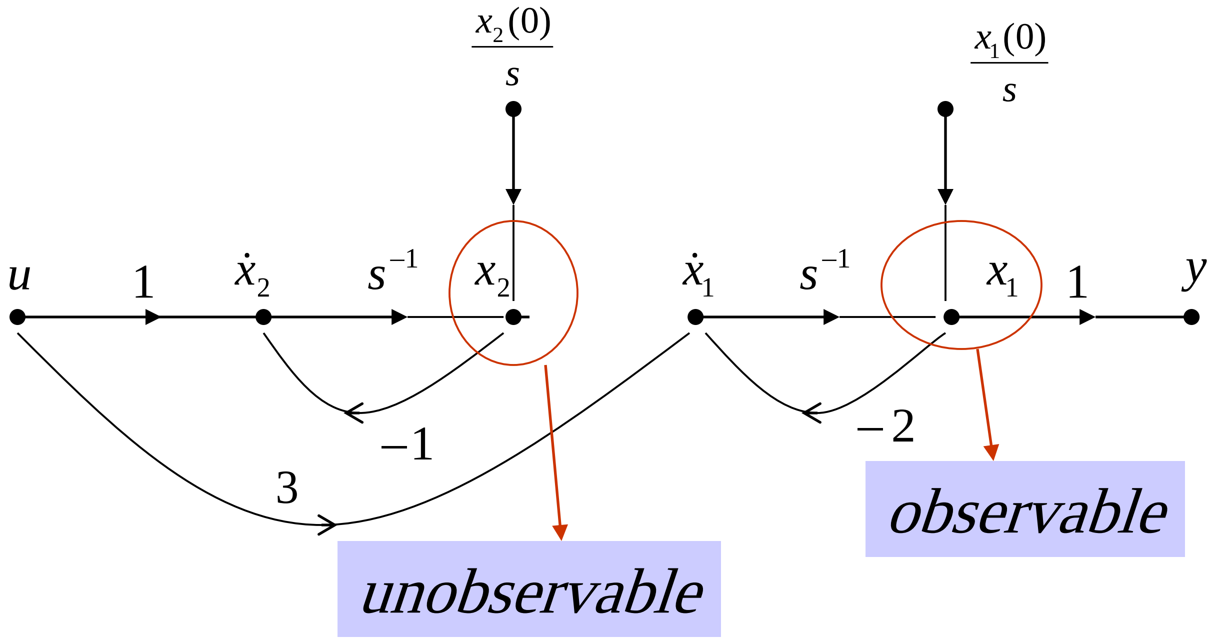
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Motivation2

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}3\\ 1\end{bmatrix}u(t)$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$



Definition

A linear system is said to be *completely controllable* if, for all initial times t_0 and all initial states $x(t_0)$, there exists some input function (or sequence for discrete systems) that drives the state vector to any final state $x(t_1)$ at some finite time $t_0 < t_1$.

Definition

A linear system is said to be *completely observable* if, for all initial times t_0, the state vector $x(t_0)$ can be determined from the output function (or sequence) $y(t_1)$, defined over a finite time $t_0 < t_1$.

Proof of controllability matrix

$$x_{k+1} = Ax_k + Bu_k$$

$$x_{k+2} = Ax_{k+1} + Bu_{k+1}$$

$$x_{k+2} = A(Ax_k + Bu_k) + Bu_{k+1} = A^2 x_k + ABu_k + Bu_{k+1}$$

$$x_{k+n} = A^n x_k + A^{n-1}Bu_k + A^{n-2}Bu_{k+1} + \cdots + ABu_{k+(n-2)} + Bu_{k+(n-1)}$$

$$x_{k+n} - A^n x_k = A^{n-1}Bu_k + A^{n-2}Bu_{k+1} + \cdots + ABu_{k+(n-2)} + Bu_{k+(n-1)}$$

$$x_{k+n} - A^n x_k = \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u_k \\ \vdots \\ u_{k+(n-2)} \\ u_{k+(n-1)} \end{bmatrix}$$

Initial condition

Proof of observability matrix

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k \cdots (1)$$

$$y_{k+1} = Cx_{k+1} + Du_{k+1}$$

$$y_{k+1} = C(Ax_k + Bu_k) + Du_{k+1} = CAx_k + CBu_k + Du_{k+1} \cdots (2)$$

$$y_{k+(n-1)} = CA^{n-1}x_k + CA^{n-2}Bu_k + CA^{n-3}Bu_{k+1} + \cdots + CBu_{k+(n-2)} + Du_{k+(n-1)} \cdots (n)$$

$$(1),(2),\cdots(n) \Rightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_k$$

$$= \underbrace{\begin{bmatrix} y_k - Du_k & y_{k+1} - CBu_k - Du_{k+1} & \cdots & \cdots & CABu_{k+(n-3)} - CBu_{k+(n-2)} - Du_{k+(n-1)} \end{bmatrix}}_{\text{Inputs \& outputs}}$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Theorem 1.3.1 *A necessary and sufficient condition of controllability of the system (1.5) on the interval* $[0, T]$, *for any* $T > 0$, *is*

$$\operatorname{rank} \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} = n.$$

Controllability matrix $U = \begin{bmatrix} B & AB & A^2B \cdots & A^{n-1}B \end{bmatrix}$

Observability matrix $V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Then: (1) controllable $rank(U) = n$

(2) observable $rank(V) = n$

Example 1

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u(t)$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$U=[B\quad AB]=\begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix}\qquad \boxed{rank[U]=1}$$

uncontrollable

$$V=\begin{bmatrix}C\\ CA\end{bmatrix}=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}\qquad \boxed{rank[V]=2}$$

observable

*The **rank** of a matrix is defined by the number of linearly independent rows and/or the number of linearly independent columns*

Example 2

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$U = [B \quad AB] = \begin{bmatrix} 3 & -6 \\ 1 & -1 \end{bmatrix} \qquad \boxed{rank[U] = 2} \qquad \textit{controllable}$$

$$V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} \qquad \boxed{rank[V] = 1} \qquad \textit{unobservable}$$

Theorem III

$$\dot{x}_c(t) = A_c x_c(t) + B_c u(t)$$

Controllable canonical form $\Rightarrow$ Controllable

Theorem IV

$$\dot{x}_o(t) = A_o x_o(t) + B_o u(t)$$

$$y(t) = C_o x_o(t)$$

Observable canonical form $\Rightarrow$ Observable *example*

$$T(s)=\frac{s+2}{(s+1)(s+2)}$$

Controllable canonical form

$$\dot{x}_c=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}x_c+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}2 & 1\end{bmatrix}x_c$$

$$U=\begin{bmatrix}B & AB\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & -3\end{bmatrix}$$

$$rank[U]=2=n$$

$$V=\begin{bmatrix}C\\ CA\end{bmatrix}=\begin{bmatrix}2 & 1\\ -2 & -1\end{bmatrix}$$

$$rank[V]=1\neq n$$

Observable canonical form

$$\dot{x}_o=\begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}x_o+\begin{bmatrix}2\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}0 & 1\end{bmatrix}x_o$$

$$U=\begin{bmatrix}B & AB\end{bmatrix}=\begin{bmatrix}2 & -2\\ 1 & -1\end{bmatrix}$$

$$rank[U]=1\neq n$$

$$V=\begin{bmatrix}C\\ CA\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & -3\end{bmatrix}$$

$$rank[V]=2=n$$

Theorem V

$$\dot{x}(t) = Jx(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Jordan form

Jordan block

$$J = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

Least row has no zero row

$$C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}$$

First column has no zero column

Example

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} b_{11} \\ b_{12} \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_{11} & c_{12} & 3 \end{bmatrix} x$$

If $b_{12} = 0$ $\Rightarrow$ uncontrollable

If $c_{11} = 0$ $\Rightarrow$ unobservable

$$\dot{x} = \left[\begin{array}{cc:c:c:ccc}
\lambda_1 & 1 & & & & & \\
 & \lambda_1 & & & & & \\
\hdashline
 & & \lambda_1 & & & & \\
\hdashline
 & & & \lambda_1 & & & \\
\hdashline
 & & & & \lambda_2 & 1 & \\
 & & & & & \lambda_2 & 1 \\
 & & & & & & \lambda_2
\end{array}\right] x + \left[\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
\hdashline
0 & 1 & 0 \\
\hdashline
0 & 0 & 1 \\
\hdashline
1 & 1 & 2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right] u \quad \begin{array}{l}
\\
\leftarrow b11 \\
\leftarrow b12 \\
\leftarrow b13 \\
\\
\\
\leftarrow b21
\end{array}$$

$$y = \begin{array}{c}
\left[\begin{array}{cc:c:c:ccc}
1 & 1 & 2 & 0 & 0 & 2 & 0 \\
1 & 0 & 1 & 2 & 0 & 1 & 1 \\
1 & 0 & 2 & 3 & 0 & 2 & 2
\end{array}\right] x \\
\begin{array}{ccccccc}
\uparrow & & \uparrow & \uparrow & \uparrow & & \\
C_{11} & & C_{12} & C_{13} & C_{21} & &
\end{array}
\end{array}$$

$$\{b_{11} \quad b_{12} \quad b_{13}\} L.I. + \quad \{b_{21}\} L.I. \Leftrightarrow$$ controllable

$$\{C_{11} \quad C_{12} \quad C_{13}\} L.I. + \quad \{C_{21}\} L.I. \Leftrightarrow$$ observable

In the previous example

$$\{b_{11} \quad b_{12} \quad b_{13}\} L.I. + \quad \{b_{21}\} L.I. \Leftrightarrow$$ controllable

$$\{C_{11} \quad C_{12} \quad C_{13}\} L.I. + \quad \{C_{21}\} L.D. \Leftrightarrow$$ unobservable

Example

$$\dot{x} = \begin{bmatrix} 2 & 1 & & & & & \\ & 2 & & & & & \\ & & 2 & & & & \\ & & & 2 & & & \\ & & & & 1 & 1 & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

L.I.

L.I.

L.I. L.D.