MENG366

Mathematical Modeling of Mechanical and Electrical Systems & Linearization

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Translational Mechanical Systems

- Basic (Idealized) Modeling Elements
- Interconnection Relationships -Physical Laws
- Derive Equation of Motion (EOM) - SDOF
- Energy Transfer
- Series and Parallel Connections
- Derive Equation of Motion (EOM) - MDOF

Key Concepts to Remember

- Three primary elements of interest
  - Mass (inertia) *M*
  - Stiffness (spring) *K*
  - Dissipation (damper) *B*
  - Usually we deal with "equivalent" *M*, *K*, *B*
    - *Distributed mass → lumped mass*
- Lumped parameters
  - Mass maintains motion (Kinetic Energy)
  - Stiffness restores motion (Potential Energy)
  - Damping eliminates motion (~~Eliminate Energy~~ ?) (Absorb Energy)

Variables

- $\boldsymbol{x}$: ***displacement*** [m]
- $\boldsymbol{v}$: ***velocity*** [m/sec]
- $\boldsymbol{a}$: ***acceleration*** [m/sec^2]
- $\boldsymbol{f}$: ***force*** [N]
- $\boldsymbol{p}$: ***power*** [Nm/sec]
- $\boldsymbol{w}$: ***work*** (energy) [Nm]

  1 [Nm] = 1 [J] (Joule)

$$v = \dot{x} = \frac{d}{dt}x$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{d}{dt}x\right) = \frac{d^2}{dt^2}x = \ddot{x}$$

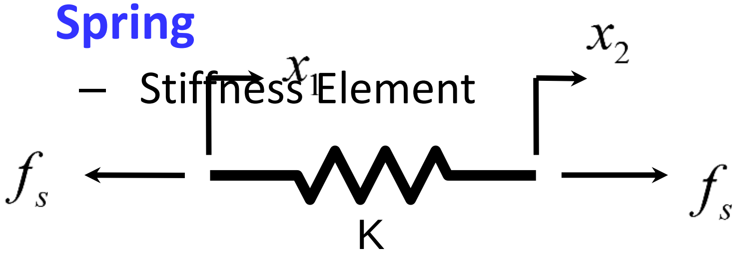
$$p = f \cdot v = f \cdot \dot{x} = \frac{d}{dt}w$$

$$w(t_1) = w(t_0) + \int_{t_0}^{t_1} p(t)\, dt$$

$$= w(t_0) + \int_{t_0}^{t_1} (f \cdot \dot{x})\, dt$$

Basic (Idealized) Modeling Elements

- **Spring**
  - Stiffness Element



$$f_S = K\underbrace{(x_2 - x_1)}_{\delta x}$$

  - Idealization
    - Massless
    - No Damping
    - Linear
  - Stores Energy

Potential Energy

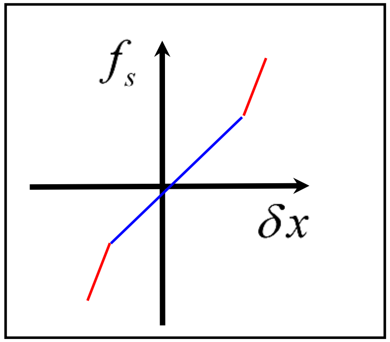
$$U = \frac{1}{2}k(\delta x)^2$$

- Reality
  - 1/3 of the spring mass may be considered into the lumped model.
  - In large displacement operation springs are *nonlinear*.

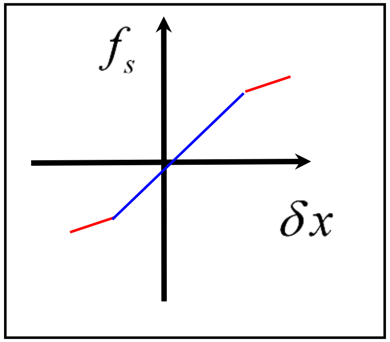
Linear spring

→ nonlinear spring

→ broken spring !!



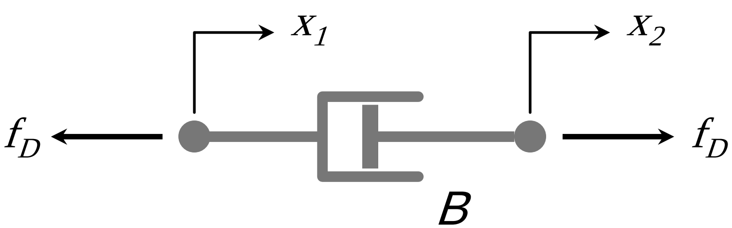
Hard Spring



Soft Spring

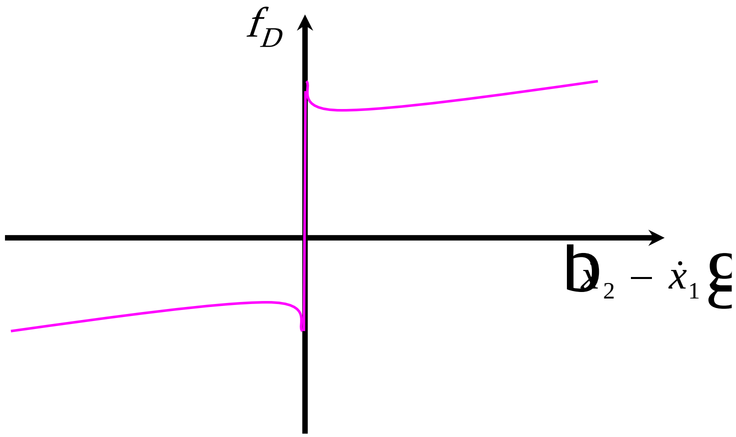
Basic Modeling Elements

- ***Damper***
  - Friction Element

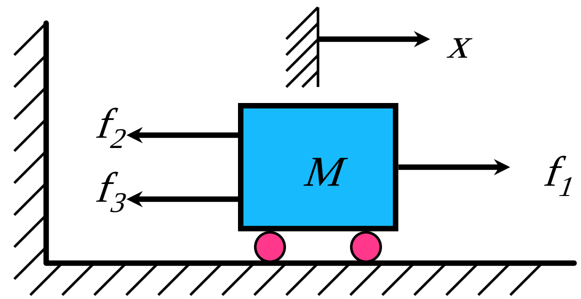


$$f_D = B(\dot{x}_2 - \dot{x}_1) = B(v_2 - v_1)$$

  - Dissipate Energy



- ***Mass***
  - Inertia Element



$$M\ddot{x} = \sum_i f_i = f_1 - f_2 - f_3$$

  - Stores Kinetic Energy

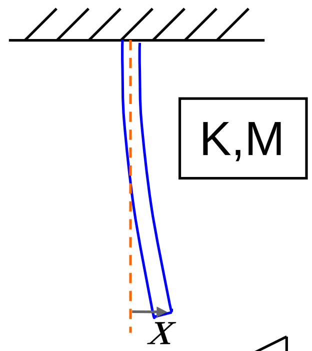
$$T = \frac{1}{2} M \dot{x}^2$$

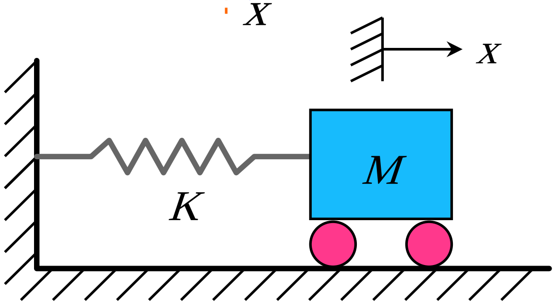
Interconnection Laws

- Newton's Second Law

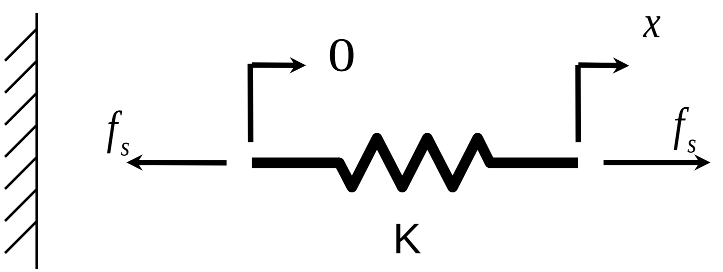
$$\frac{d}{dt}\underbrace{\left(M\,v\right)}_{\text{Linear Momentum}} = M\,\ddot{x} = \sum_i f_{EXTi}$$

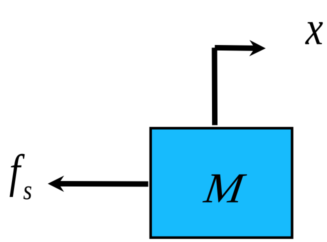
Lumped Model of a Flexible Beam





- Newton's Third Law
  - Action & Reaction Forces





$$f_s = K\left(x-0\right) = Kx$$

Massless spring

$$M\,\ddot{x} = -f_s$$

$$M\,\ddot{x} = -Kx$$

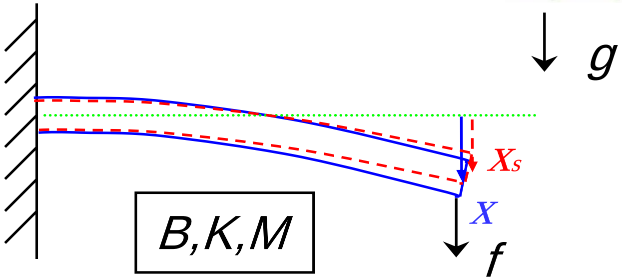
$$\boxed{M\,\ddot{x} + Kx = 0}$$

E.O.M.

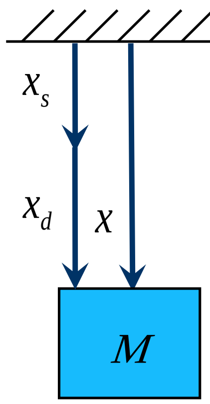
Modeling Steps

- ***Understand System Function, Define Problem, and Identify Input/Output Variables***
- ***Draw Simplified Schematics Using Basic Elements***
- ***Develop Mathematical Model (Diff. Eq.)***
  – Identify reference point and positive direction.
  – Draw Free-Body-Diagram (FBD) for each basic element.
  – Write Elemental Equations as well as Interconnecting Equations by applying physical laws. (*Check: # eq = # unk)*
  – Combine Equations by eliminating intermediate variables.
- ***Validate Model by Comparing Simulation Results with Physical Measurements***

Single Degree of Freedom (SDOF) System



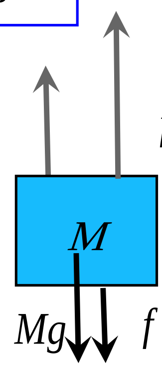
- ***Define Problem*** — *The motion of the object*
- ***Input*** — f
- ***Output*** — x
- ***Develop Mathematical Model (Diff. Eq.)***
  - Identify reference point and positive direction.

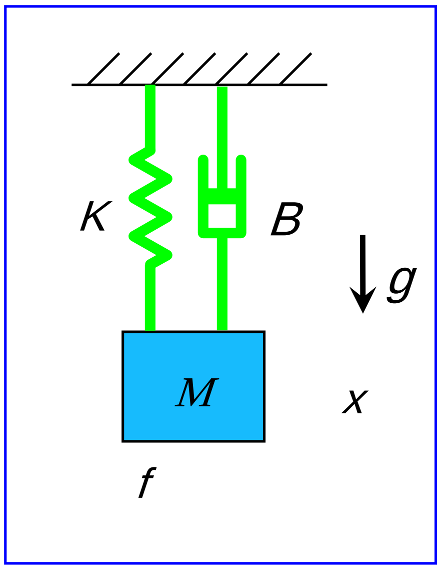


$$x = x_s + x_d$$
$$Kx_s = Mg$$

$$f_s = Kx \qquad f_d = B\dot{x}$$

  - Draw Free-Body-Diagram (FBD)





  - Write Elemental Equations

$$M\ddot{x} = -B\dot{x} - Kx + Mg + f$$

$$M\ddot{x} + B\dot{x} + Kx = Mg + f$$

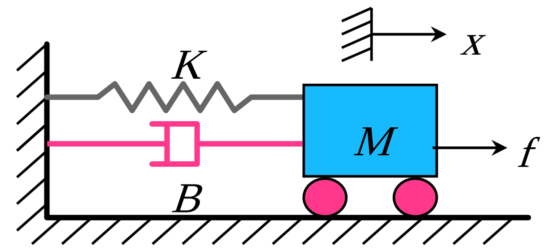
$$M\ddot{x}_u + B\dot{x}_u + Kx_u = Mg + f$$ *From the undeformed position*

$$M\ddot{x}_d + B\dot{x}_d + Kx_d = f$$ *From the deformed (static equilibrium) position*

- ***Validate Model by Comparing Simulation Results with Physical Measurement***

Energy Distribution

- ***EOM of a simple Mass-Spring-Damper System***



$$M\ddot{x} + B\dot{x} + Kx = f(t)$$

Contribution of Inertia	Contribution of the Damper	Contribution of the Spring	Total Applied Force

We want to look at the energy distribution of the system. How should we start ?

- ***Multiply the above equation by the velocity term v :*** ⇐*What have we done ?*

$$M\ddot{x}\cdot\dot{x} + B\dot{x}\cdot\dot{x} + Kx\cdot\dot{x} = f(t)\cdot\dot{x}$$

- ***Integrate the second equation w.r.t. time:*** ⇐*What are we doing now ?*

$$\underbrace{\int_{t_0}^{t_1} M\ddot{x}\cdot\dot{x}\,dt}_{\Delta KE} + \underbrace{\int_{t_0}^{t_1} B\dot{x}\cdot\dot{x}\,dt}_{\int_{t_0}^{t_1} B\dot{x}^2 dt \,\geq\, 0} + \underbrace{\int_{t_0}^{t_1} Kx\cdot\dot{x}\,dt}_{\Delta PE} = \underbrace{\int_{t_0}^{t_1} f(t)\cdot v\,dt}_{W}$$

ΔKE ⇓ Change of kinetic energy

$\int_{t_0}^{t_1} B\dot{x}^2 dt \geq 0$ ⇓ Energy dissipated by damper

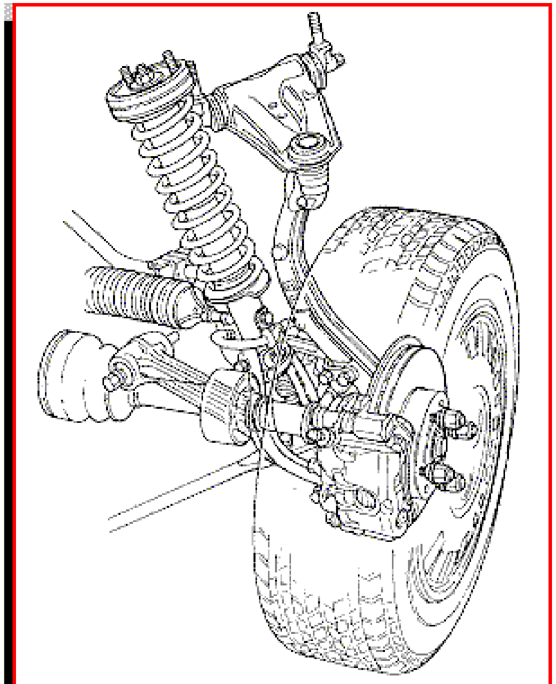
ΔPE ⇓ Change of potential energy

W: Total work done by the applied force $f(t)$ from time t_0 to t_1

SDOF Suspension (Example)

- ***Suspension System***

  Minimize the effect of the surface roughness of the road on the drivers comfort.



Double Wishbone Suspension

– ***Simplified Schematic (neglecting tire model)***

From the "absolute zero"

$$M\ddot{x} + B\left(\dot{x} - \dot{x}_p\right) + K\left(x - x_p\right) = -Mg$$

From the path

$$M\ddot{x}_{op} + B\dot{x}_{op} + Kx_{op} = -Mg - M\dot{x}_p$$

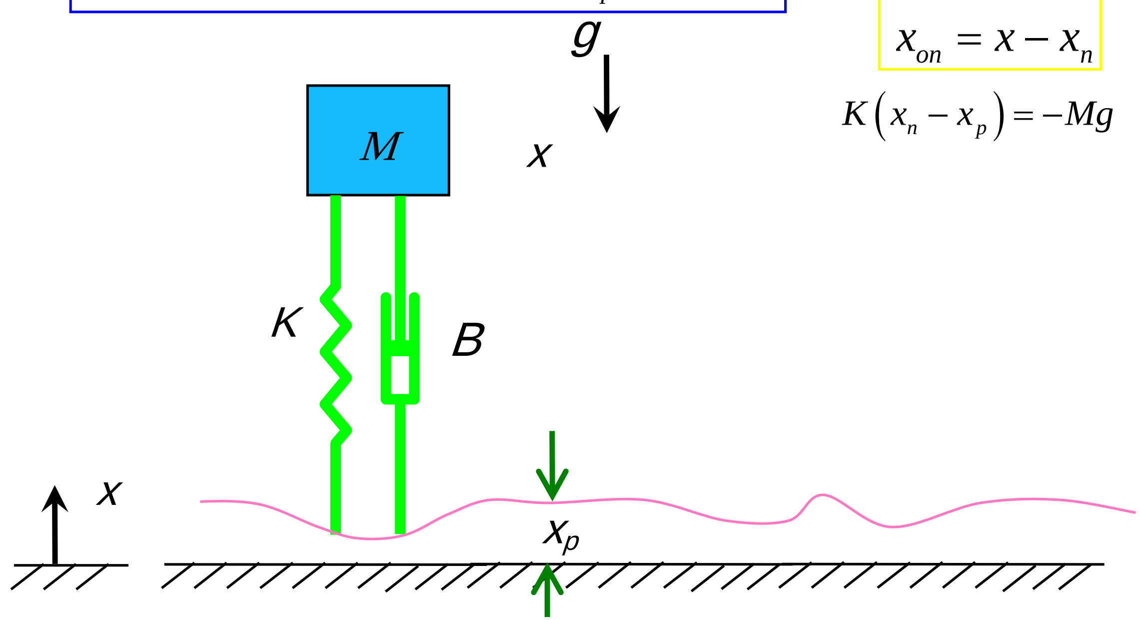
$$x_{op} = x - x_p$$

From nominal position

$$M\ddot{x}_{on} + B\dot{x}_{on} + Kx_{on} = -M\ddot{x}_p$$

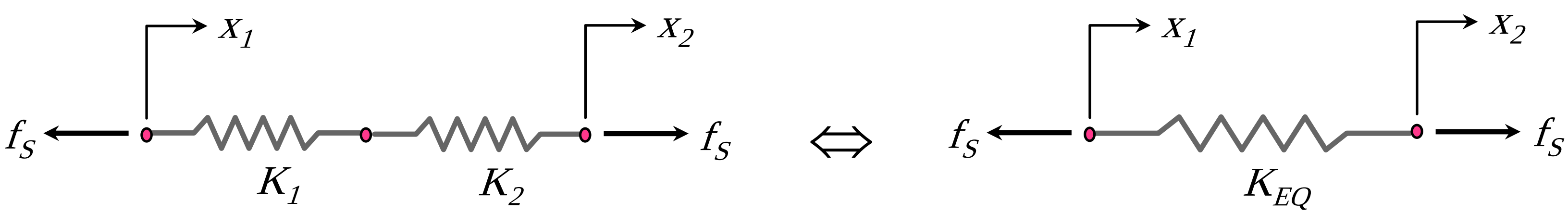
$$x_{on} = x - x_n$$

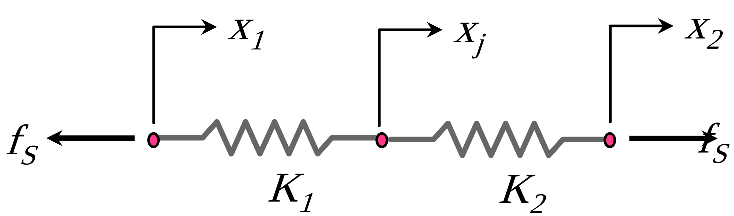
$$K\left(x_n - x_p\right) = -Mg$$



Series Connection

- Springs in Series





$$K_1\left(x_j - x_1\right) = K_2\left(x_2 - x_j\right)$$

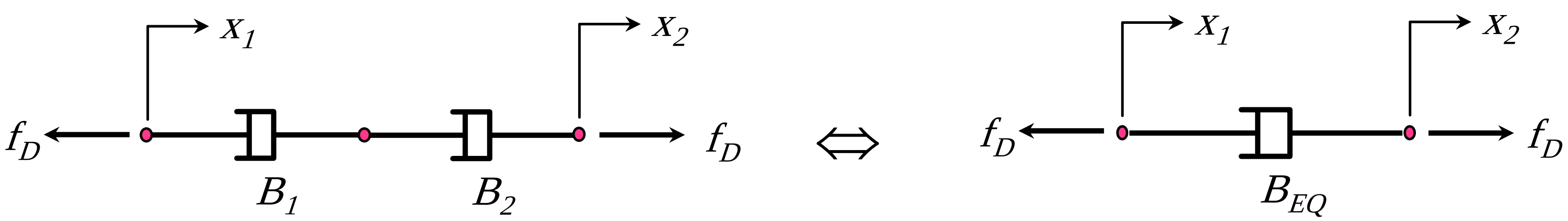
$$x_j = \frac{1}{K_1 + K_2}\left[K_2 x_2 + K_1 x_1\right]$$

$$f_s = K_1\left(x_j - x_1\right) = K_1\left\{\underbrace{\frac{1}{K_1 + K_2}\left[K_2 x_2 + K_1 x_1\right]}_{x_j} - x_1\right\}$$

$$f_s = \underbrace{\frac{K_1 K_2}{K_1 + K_2}}_{K_{eq}}\left[x_2 - x_1\right]$$

Series Connection

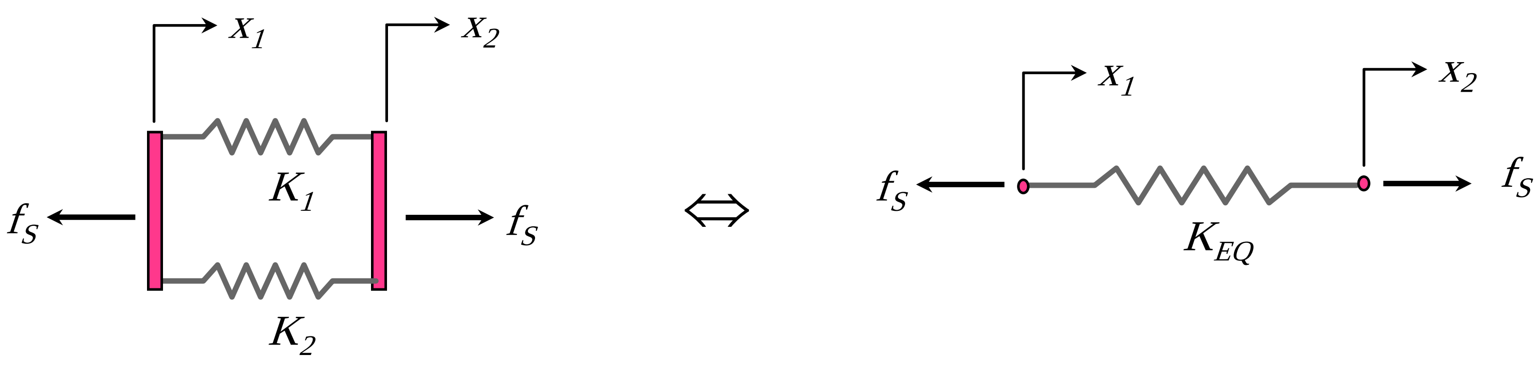
- Dampers in Series



$$f_d = \underbrace{\frac{B_1 B_2}{B_1 + B_2}}_{B_{eq}} \left[\dot{x}_2 - \dot{x}_1 \right]$$

Parallel Connection

- Springs in Parallel

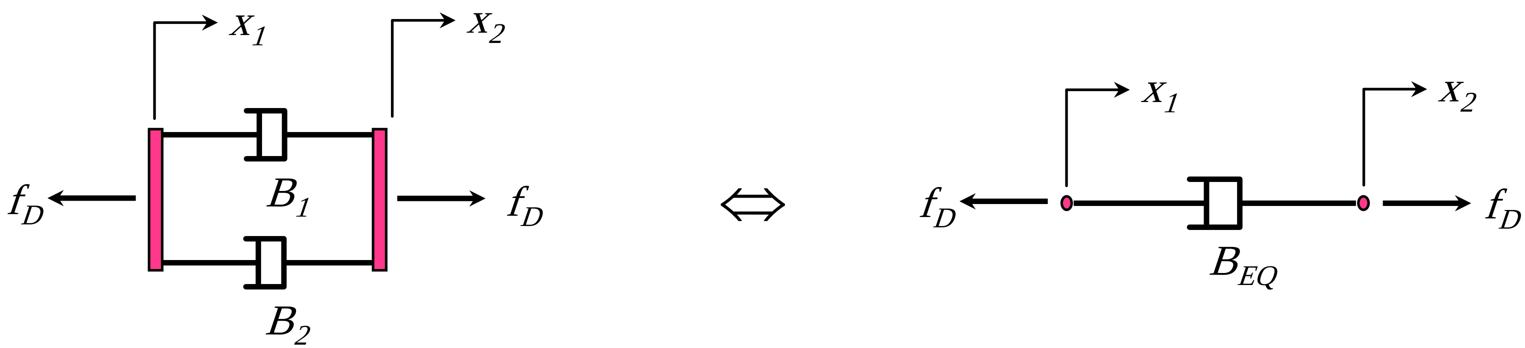


$$f_s = K_1(x_2 - x_1) + K_2(x_2 - x_1)$$

$$f_s = \underbrace{(K_1 + K_2)}_{K_{eq}}(x_2 - x_1)$$

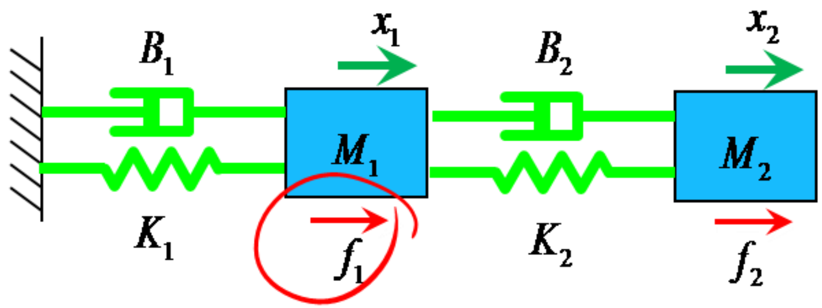
Parallel Connection

Dampers in Parallel

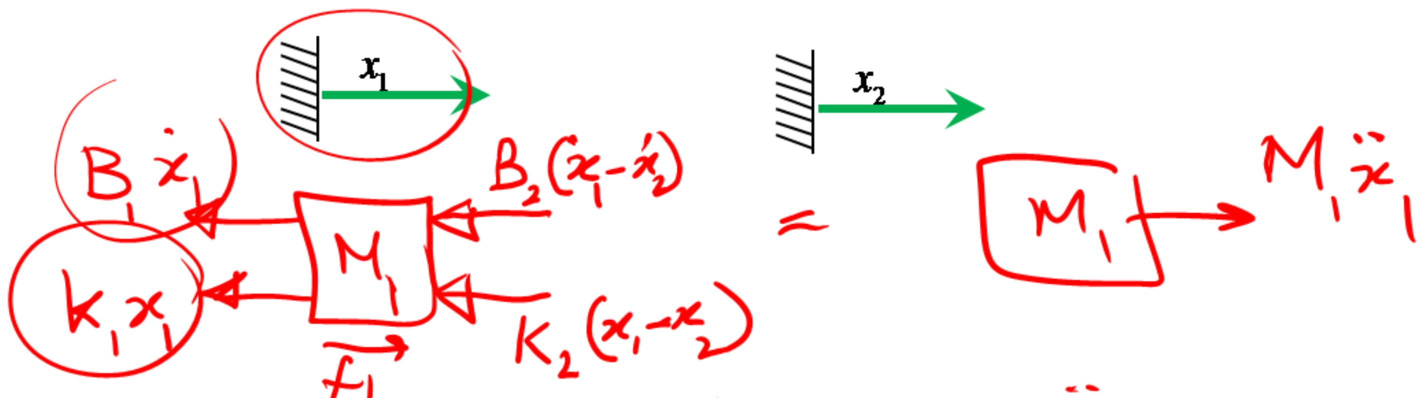


$$f_s = \underbrace{\left(B_1 + B_2\right)}_{K_{eq}}\left(\dot{x}_2 - \dot{x}_1\right)$$

Two Degree of Freedom (TDOF) System



For x_1:



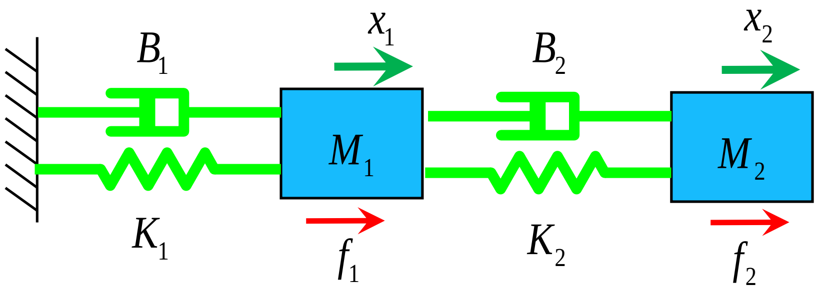
$$-B_1\dot{x}_1 - K_1x_1 - B_2(\dot{x}_1-\dot{x}_2) - K_2(x_1-x_2) + f_1 = M_1\ddot{x}_1$$

$$-B_1\dot{x}_1 - K_1x_1 - B_2\dot{x}_1 + B_2\dot{x}_2 - K_2x_1 + K_2x_2 + f_1 = M_1\ddot{x}_1$$

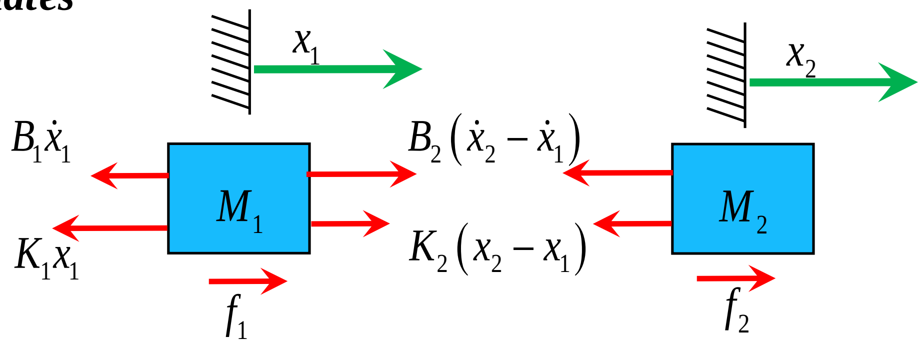
$$M_1\ddot{x}_1 + (B_1+B_2)\dot{x}_1 + (K_1+K_2)x_1 - B_2\dot{x}_2 - K_2x_2 = f_1$$

Two Degree of Freedom (TDOF) System

- ***DOF = 2***



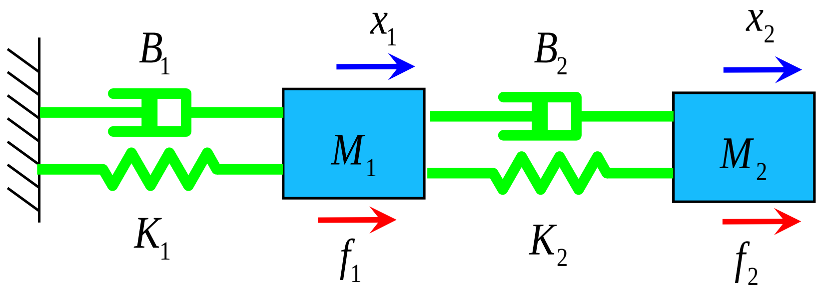
- ***Absolute coordinates***
- ***FBD***



- ***Newton's law***

$$M_1\ddot{x}_1 = -B_1\dot{x}_1 - K_1x_1 + B_2(\dot{x}_2-\dot{x}_1) + K_2(x_2-x_1) + f_1(t)$$
$$M_2\ddot{x}_2 = -B_2(\dot{x}_2-\dot{x}_1) - K_2(x_2-x_1) + f_2(t)$$

Two Degree of Freedom (TDOF) System



Static coupling

- ***Absolute coordinates***

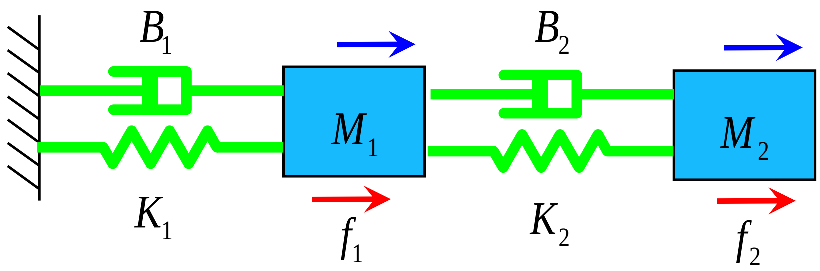
$$M_1\ddot{x}_1 + (B_1 + B_2)\dot{x}_1 + (K_1 + K_2)x_1 - B_2\dot{x}_2 - K_2x_2 = f_1(t)$$
$$M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2x_2 - B_2\dot{x}_1 - K_2x_1 = f_2(t)$$

- ***Relative coordinates***

$$M_1\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - B_2\dot{x}_{21} - K_2x_{21} = f_1(t)$$
$$M_2\ddot{x}_1 + M_2\ddot{x}_{21} + B_2\dot{x}_{21} + K_2x_{21} = f_2(t)$$

Dynamic coupling

Two DOF System – Matrix Form of EOM



- ***Absolute coordinates***

Mass matrix

Damping matrix

Output vector

Input vector

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} B_1 + B_2 & -B_2 \\ -B_2 & B_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

Stiffness matrix

SYMMETRIC

- ***Relative coordinates***

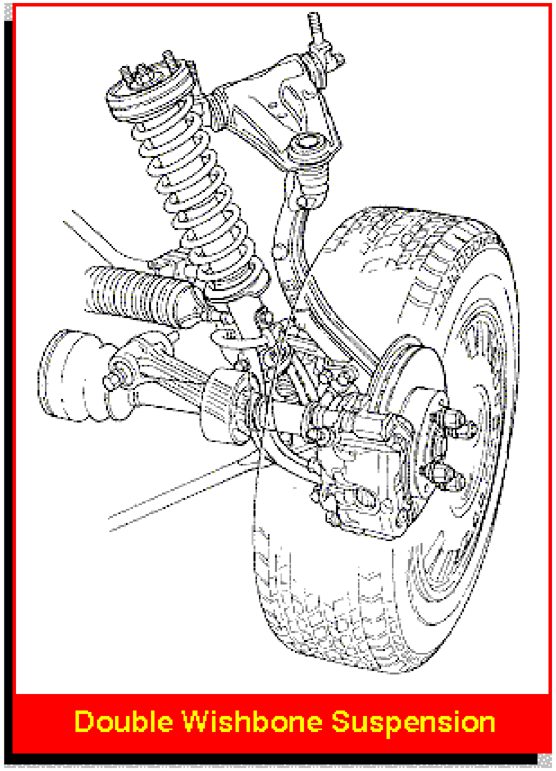
$$\begin{bmatrix} M_1 & 0 \\ M_2 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_{21} \end{Bmatrix} + \begin{bmatrix} B_1 & -B_2 \\ 0 & B_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_{21} \end{Bmatrix} + \begin{bmatrix} K_1 & -K_2 \\ 0 & K_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_{21} \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

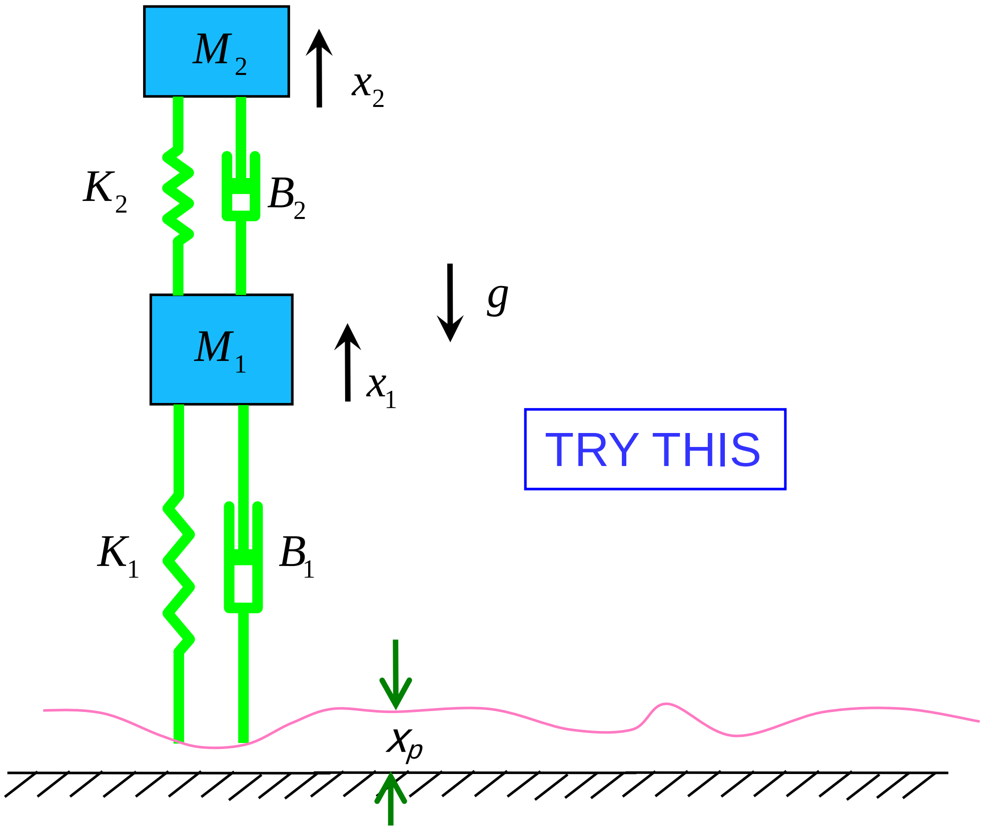
NON-SYMMETRIC

MDOF Suspension

Example 1

- ***Suspension System***
  - Simplified Schematic (with tire model)

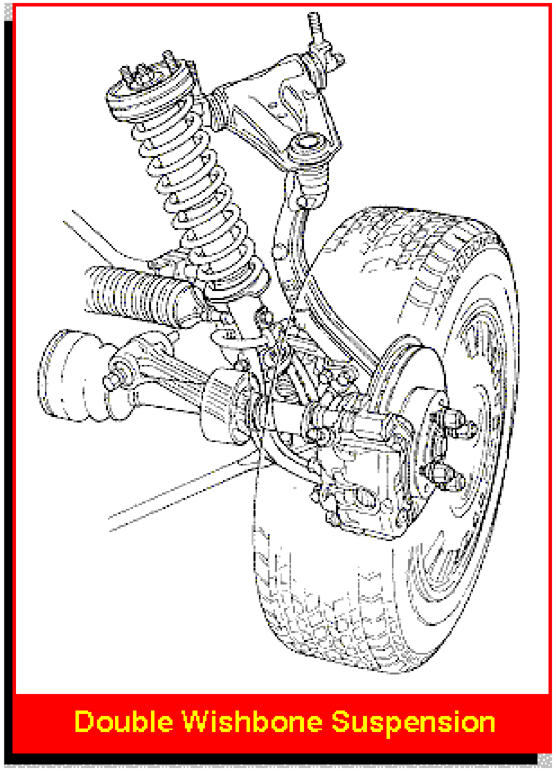




TRY THIS

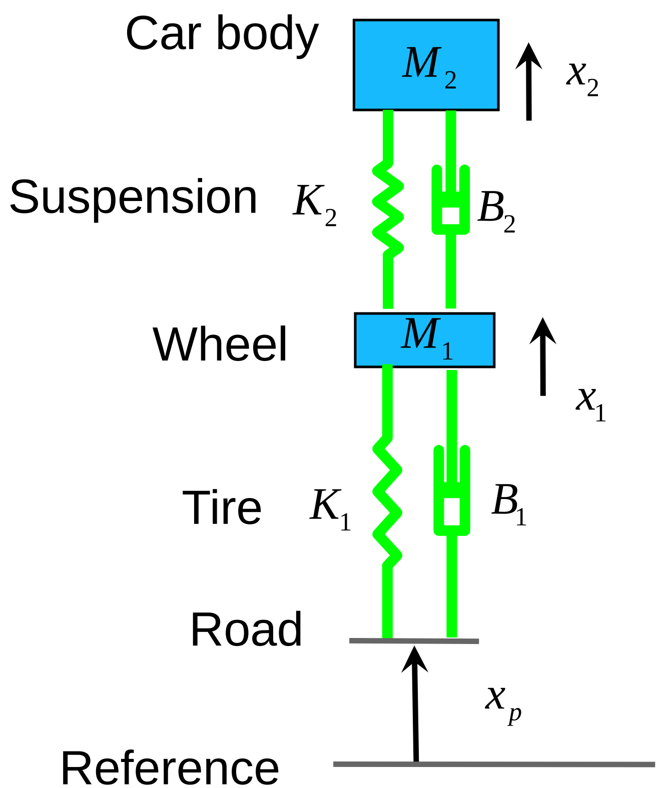
MDOF Suspension

- ***Suspension System***



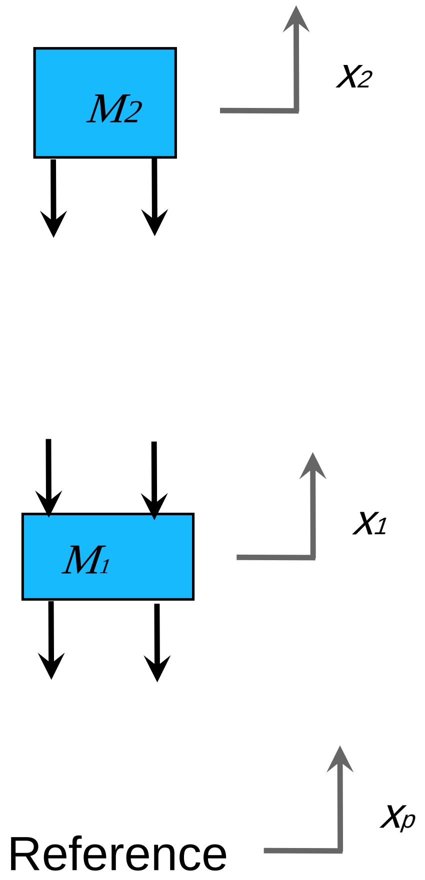
– Simplified Schematic (with tire model)

Assume ref. is when springs are Deflected by weights



MDOF Suspension

– *Draw FBD*



– *Apply Newton's 2nd Laws*

$$\Rightarrow \begin{aligned} &M_2\ddot{x}_2 + B_2\dot{x}_2 - B_2\dot{x}_1 + K_2x_2 - K_2x_1 = 0 \\ &M_1\ddot{x}_1 - B_2\dot{x}_2 + \left(B_2 + B_1\right)\dot{x}_1 - K_2x_2 + \left(K_2 + K_1\right)x_1 = B_1\dot{x}_p + K_1x_p \end{aligned}$$

MDOF Suspension

– ***Matrix Form***

$$M_2\ddot{x}_2 + B_2\dot{x}_2 - B_2\dot{x}_1 + K_2x_2 - K_2x_1 = 0$$
$$M_1\ddot{x}_1 - B_2\dot{x}_2 + (B_2 + B_1)\dot{x}_1 - K_2x_2 + (K_2 + K_1)x_1 = B_1\dot{x}_p + K_1x_p$$

Define vector $x = \begin{Bmatrix} x_2 \\ x_1 \end{Bmatrix}$

$$\begin{bmatrix} M_2 & 0 \\ 0 & M_1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_1 \end{Bmatrix} + \begin{bmatrix} B_2 & -B_2 \\ -B_2 & B_2 + B_1 \end{bmatrix}\begin{Bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{Bmatrix} + \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 + K_1 \end{bmatrix}\begin{Bmatrix} x_2 \\ x_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ B_1\dot{x}_p + K_1x_p \end{Bmatrix}$$

Mass matrix

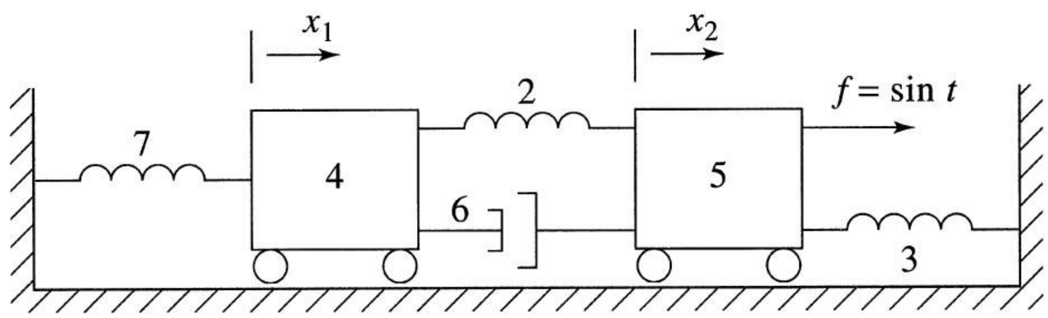
Damping matrix

Stiffness matrix

Input Vector

Two Degree of Freedom (TDOF) System

POP. Qiuz



(a)

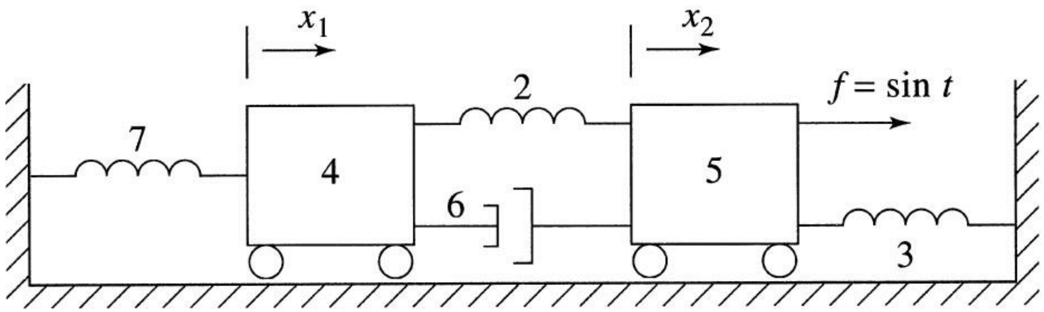
(b)

$$7x_1 + 4\frac{d^2x_1(t)}{dt^2} + 2(x_1 - x_2) + 6\frac{d(x_1 - x_2)}{dt} = 0$$

$$2(x_2 - x_1) + 6\frac{d(x_2 - x_1)}{dt} + 5\frac{d^2x_2}{dt^2} + 3x_2 = \sin t$$

Two Degree of Freedom (TDOF) System

Solution:



(a)

(b)

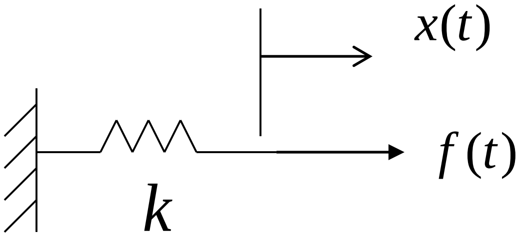
$$7x_1 + 4\frac{d^2x_1(t)}{dt^2} + 2(x_1 - x_2) + 6\frac{d(x_1 - x_2)}{dt} = 0$$

$$2(x_2 - x_1) + 6\frac{d(x_2 - x_1)}{dt} + 5\frac{d^2x_2}{dt^2} + 3x_2 = \sin t$$

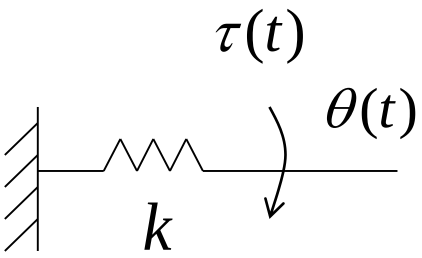
Translational mechanical components

Rotational mechanical components

spring



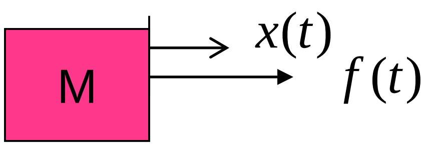
$$f(t) = kx(t)$$



$$\tau(t) = k\theta(t)$$

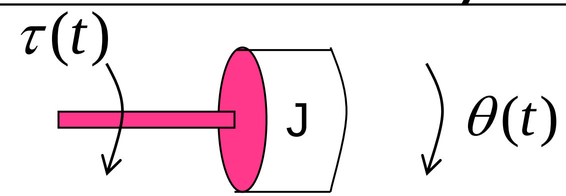
Rotational Mechanical Systems

Translational mechanical components



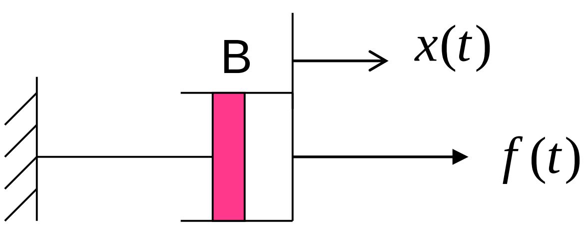
$$f(t) = Ma = Mx''(t)$$

Rotational mechanical components

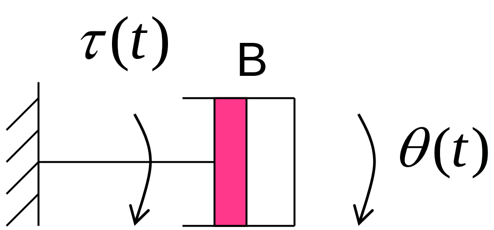


$$\tau(t) = J\theta''(t)$$

Viscous friction (linear)



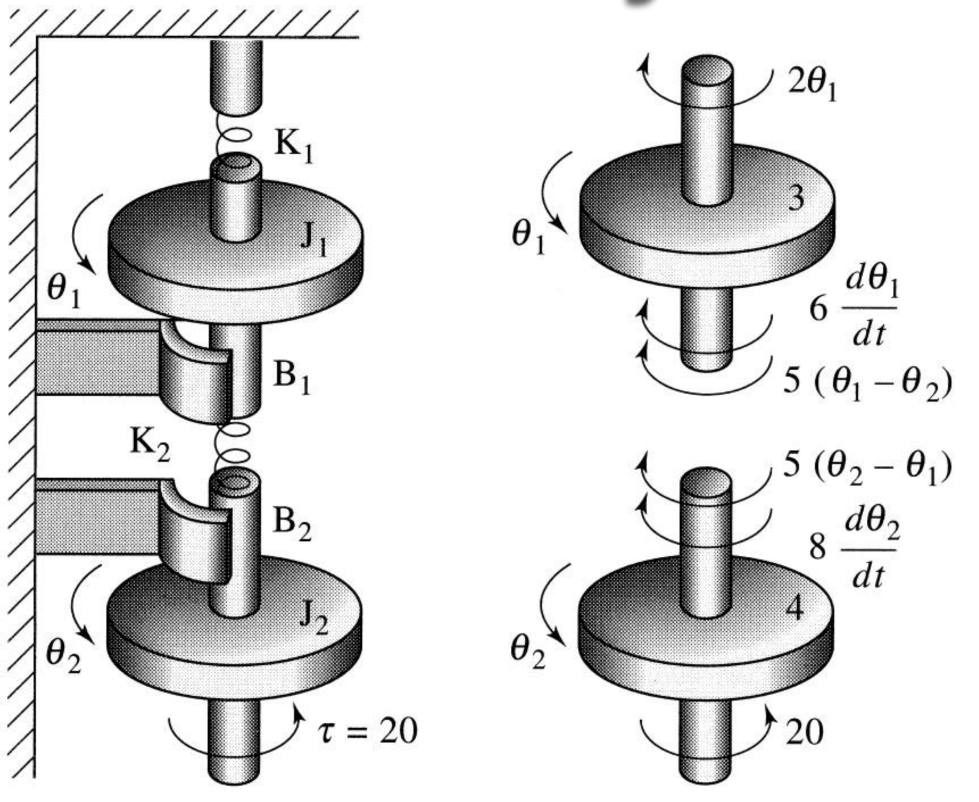
$$f(t) = Bv(t) = Bx'(t)$$



$$\tau(t) = B\theta'(t)$$

Mechanical Systems

Example 2



(a) (b)

$K_1 = 2$

$K_2 = 5$

$B_1 = 6$

$B_2 = 8$

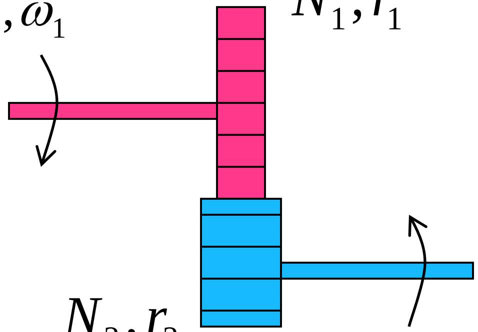
$J_1 = 3$

$J_2 = 4$

$$\begin{cases} 3\dfrac{d^2\theta_1}{dt^2} + 2\theta_1 + 5(\theta_1 - \theta_2) + 6\dfrac{d\theta_1}{dt} = 0 \\ 4\dfrac{d^2\theta_2}{dt^2} + 5(\theta_2 - \theta_1) + 8\dfrac{d\theta_2}{dt} = 20 \end{cases}$$

Mechanical Systems

Gear train



(1) $$\frac{r_1}{r_2} = \frac{N_1}{N_2}$$

$\therefore r \propto N$

(2) $$r_1\theta_1 = r_2\theta_2$$

$\therefore S_1 = S_2$

(3) $$\tau_1\theta_1 = \tau_2\theta_2$$

no energy loss

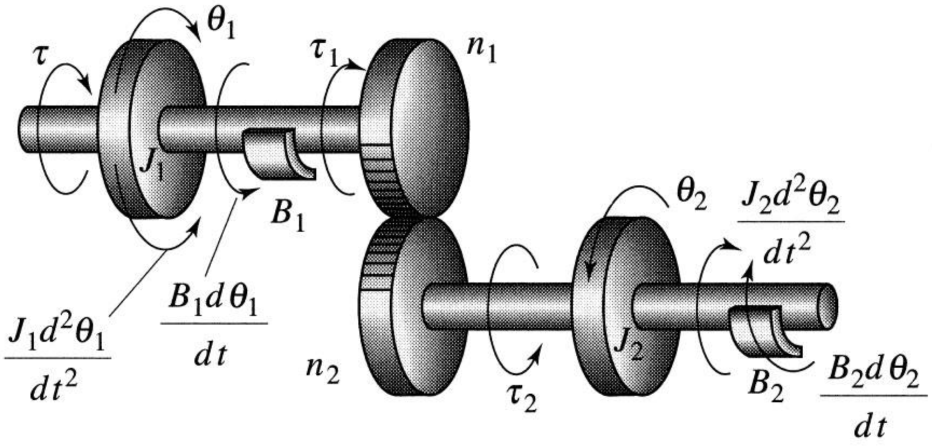
(4) $$r_1\omega_1 = r_2\omega_2$$

$\therefore S_1 = S_2$

$$\frac{N_1}{N_2} = \frac{r_1}{r_2} = \frac{\tau_1}{\tau_2} = \frac{\theta_2}{\theta_1} = \frac{\omega_2}{\omega_1}$$

Mechanical Systems

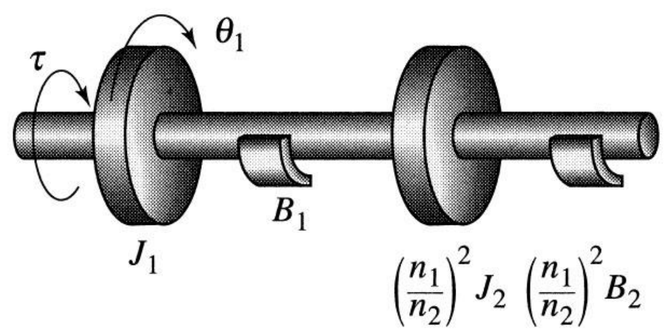
Example 3



(a)

$$\tau = J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + \tau_1 \cdots\cdots \quad (1)$$

$$\tau_2 = J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 \cdots\cdots \quad (2)$$



(b)

$$\tau_2 \quad \Rightarrow \quad \frac{N_2}{N_1}\tau_1$$

$$\ddot{\theta}_2 \quad \Rightarrow \quad \frac{N_1}{N_2}\ddot{\theta}_1$$

$$\dot{\theta}_2 \quad \Rightarrow \quad \frac{N_1}{N_2}\dot{\theta}_1$$

Electrical Systems

- ***Basic Modeling Elements***
- ***Interconnection Relationships***
- ***Derive Input/Output Models***

Variables

- ***q*** : charge **[C] (Coulomb)**
- ***i*** : current **[A]**
- ***e*** : voltage **[V]**
- ***R*** : resistance **[Ω]**
- ***C*** : capacitance **[Farad]**
- ***L*** : inductance **[H] (Henry)**
- ***p*** : power **[Watt]**
- ***w*** : work (energy) **[J]**

  1 [J] (Joule) = 1 [V-A-sec]

$$\frac{d}{dt}q = i$$

$$q(t_1) = q(t_0) + \int_{t_0}^{t_1} i(t)\, dt$$

$$p = e \cdot i$$

$$w(t_1) = w(t_0) + \int_{t_0}^{t_1} p(t)\, dt$$

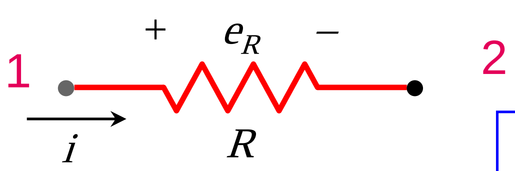
$$= w(t_0) + \int_{t_0}^{t_1} (e \cdot i)\, dt$$

Basic Modeling Elements

- **Resistor**
  - <u>Ohms Law</u>

    Voltage across is proportional to the through current.

$$e_{12} = e_1 - e_2 = e_R = R\,i \quad \Leftrightarrow \quad i = \frac{1}{R} e_R$$



Static relation

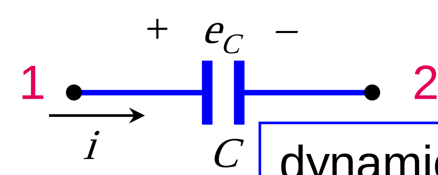
  - Dissipates energy through heat.

$$p = R\,i^2 = \frac{1}{R} e^2$$

  - Analogous to friction elements in mechanical systems, e.g. dampers

- ***Capacitor***
  - Charge collected is proportional to the voltage across.
  - Current is proportional to the rate of change of the voltage across.

$$q = C\,e_C \quad \Leftrightarrow \quad i = C\left(\frac{d}{dt} e_C\right) = C\left(\frac{d}{dt} e_{12}\right)$$



dynamic relation

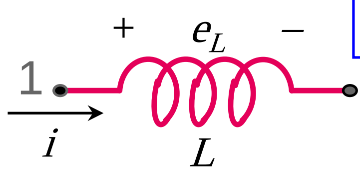
  - Energy supplied is stored in its electric field and can affect future circuit response.
  - Steady-state response: *i=0,* Open Circuit

Basic Modeling Elements

- ***Inductor***
  - Voltage across is proportional to the rate of the change of the through current.

$$e_{12} = e_L = L\left(\frac{d}{dt}i\right)$$

dynamic relation

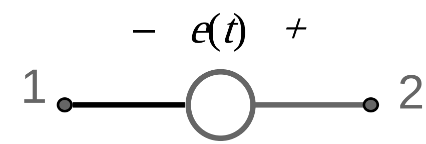


  - Energy supplied is stored in its magnetic field.

$$w = \frac{1}{2}L\,i^2$$

  - Steady-state response: *e=0,* Short Circuit

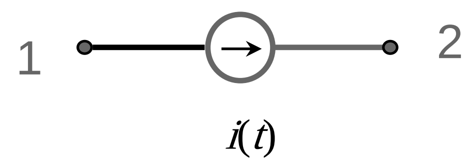
- **Voltage Source**
  - Maintain specified voltage across two points, regardless of the required current.



$$e_{12} = e_1 - e_2 = -e(t)$$

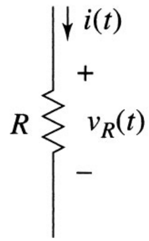
Static relation

- **Current Source**
  - Maintain specified current, regardless of the required voltage.



Basic Modeling Elements

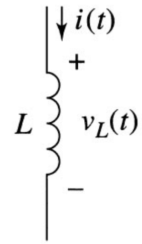
Resistor



$$v_R(t) = Ri(t)$$

$$i(t) = \frac{1}{R} v_R(t)$$

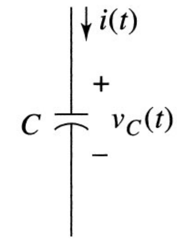
Inductor



$$v_L(t) = L\frac{di}{dt}$$

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)dt$$

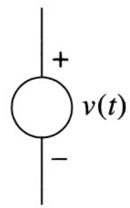
Capacitor



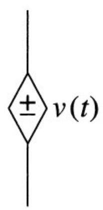
$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

$$i(t) = C\frac{dv_C}{dt}$$

Voltage Source

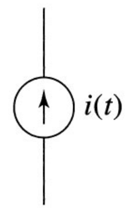


$v(t)$ a given function of time

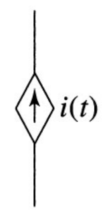


$v(t)$ expressed in terms of other network voltages or currents

Current Source



$i(t)$ a given function of time



$i(t)$ expressed in terms of other network voltages or currents

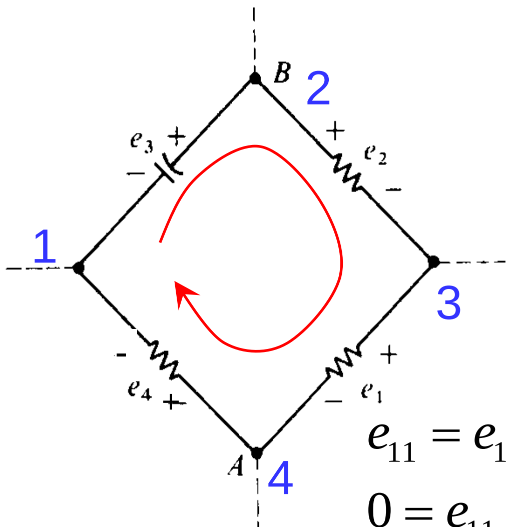
Interconnection Laws

- **Kirchhoff's Voltage Law (loop law)**
  - The total voltage drop along any *closed loop* in the circuit is zero.

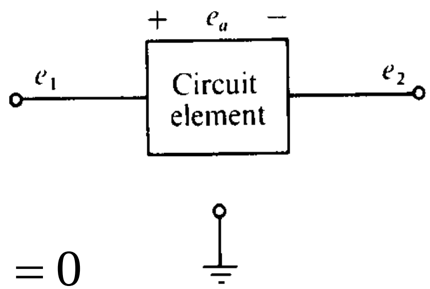
$$\sum_{\text{Closed Loop}} e_j = 0$$

- **Kirchhoff’s Current Law (node law)**
  - The algebraic sum of the currents at any node in the circuit is zero.

$$\sum_{\text{Any Node}} i_j = 0$$



$$e_a = e_{12} = e_1 - e_2$$

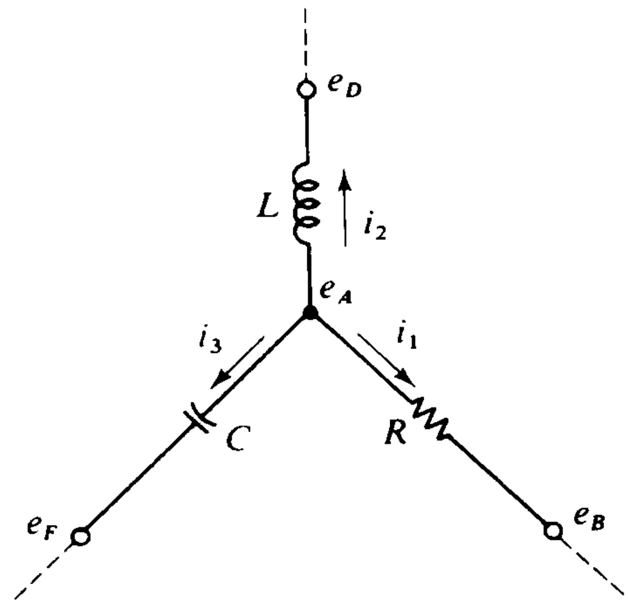


$$e_{11} = e_1 - e_1 = 0$$

$$0 = e_{11} = e_{12} + e_{23} + e_{34} + e_{41}$$

$$-e_3 + e_2 + e_1 + e_4 = 0$$

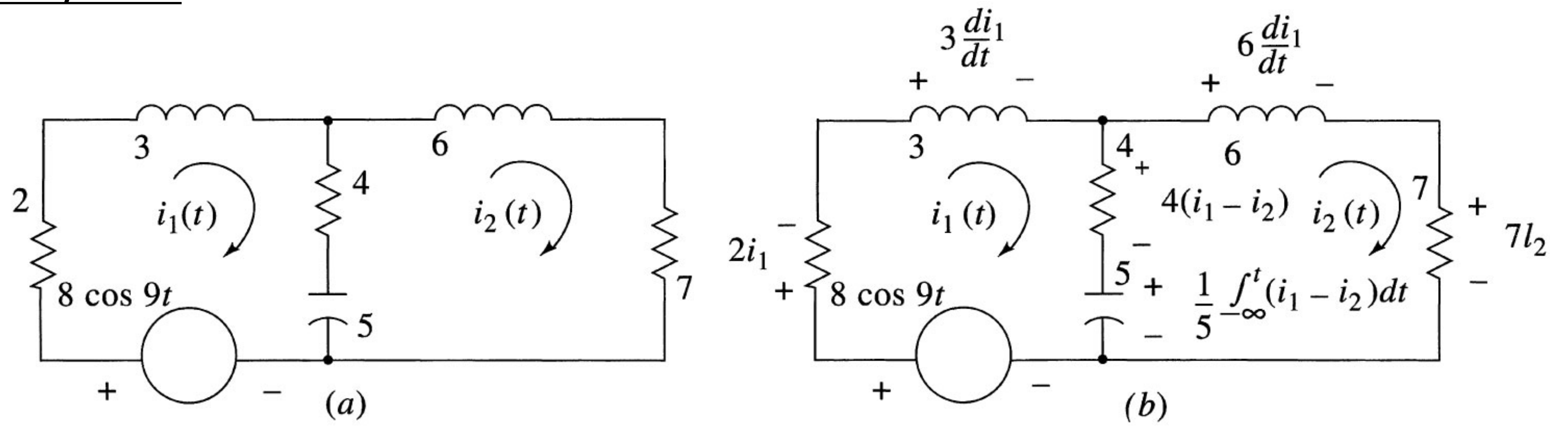
e_{12} e_{23} e_{34} e_{41}



$$-i_1 - i_2 - i_3 = 0$$

Basic Modeling Elements

Example 4



$$loop1 \quad 2i_1(t) + 3\frac{di_1(t)}{dt} + 4(i_1(t) - i_2(t)) + \frac{1}{5}\int_0^5 (i_1(t) - i_2(t))dt = 8\cos 9t$$

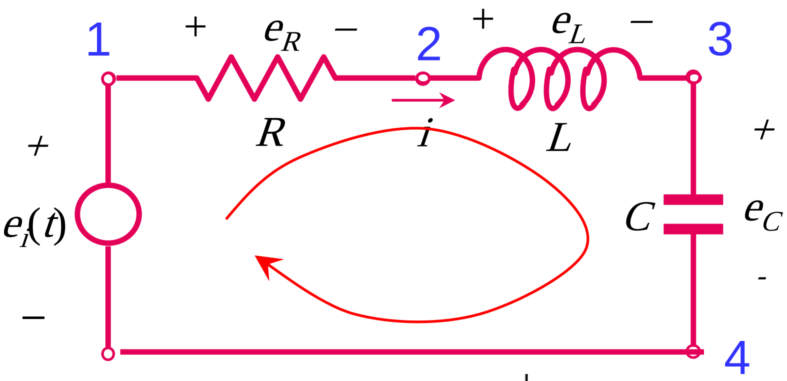
$$loop2 \quad \frac{1}{5}\int_0^t (i_2(t) - i_1(t))dt + 4(i_2(t) - i_1(t)) + 6\frac{di_2(t)}{dt} + 7i_2(t) = 0$$

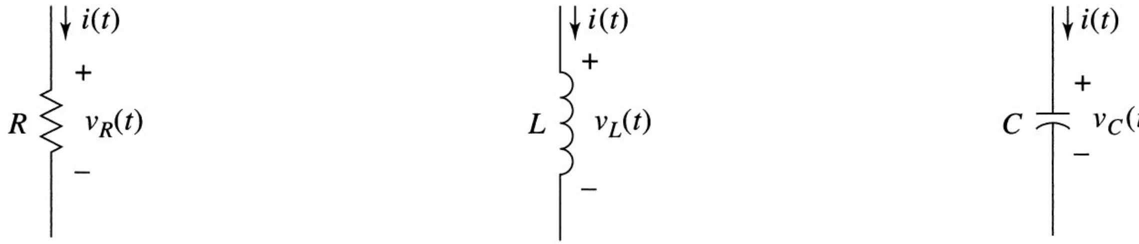
Modeling Steps

- ***Understand System Function and Identify Input/Output Variables***
- ***Draw Simplified Schematics Using Basic Elements***
- ***Develop Mathematical Model***
  - Label Each Element and the Corresponding Voltages.
  - Label Each Node and the Corresponding Currents.
  - Apply Interconnection Laws.
  - Check that the Number of Unknown Variables equals the Number of Equations
  - Eliminate Intermediate Variables to Obtain Standard Forms.

In Class Exercise

Derive the I/O model for the following circuit. Let voltage $e_i(t)$ be the input and the voltage across the capacitor be the output.

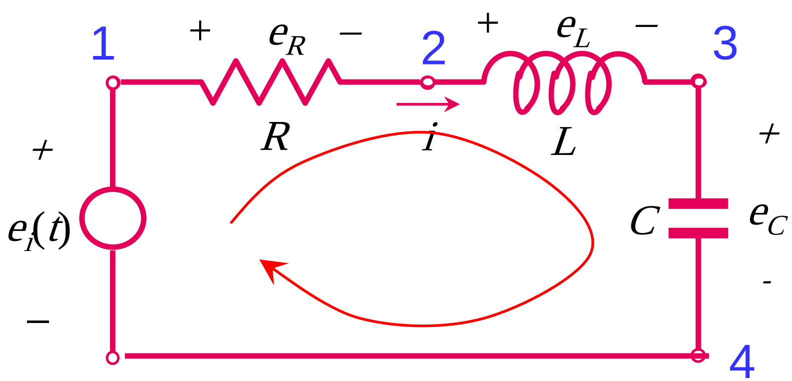




Resistor	Inductor	Capacitor
$v_R(t) = Ri(t)$	$v_L(t) = L\frac{di}{dt}$	$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$
$i(t) = \frac{1}{R}v_R(t)$	$i(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)dt$	$i(t) = C\frac{dv_C}{dt}$

In Class Exercise

Derive the I/O model for the following circuit. Let voltage $e_i(t)$ be the input and the voltage across the capacitor be the output.



Element Laws:

$$e_{12} = e_R = iR \qquad e_{23} = e_L = L\left(\frac{d}{dt}i\right)$$

$$i = C\left(\frac{d}{dt}e_{34}\right) \qquad e_{41} = -e_i(t)$$

Kirchhoff's Loop Law:

$$e_{12} + e_{23} + e_{34} + e_{41} = 0$$

No. of Unknowns:

$$e_{12}, \quad e_{23}, \quad y = e_{34}, \quad e_{41}, \quad i$$

Simplify

$$\begin{cases} i = C\dfrac{de_{34}}{dt} \\ iR + L\dfrac{di}{dt} + e_{34} - e_i = 0 \end{cases}$$

I/O Model:

$$LC\frac{d^2e_{34}}{dt^2} + RC\frac{de_{34}}{dt} + e_{34} = e_i$$

How to get I/O model by concept of complex impedance ?

Mechanical translational system

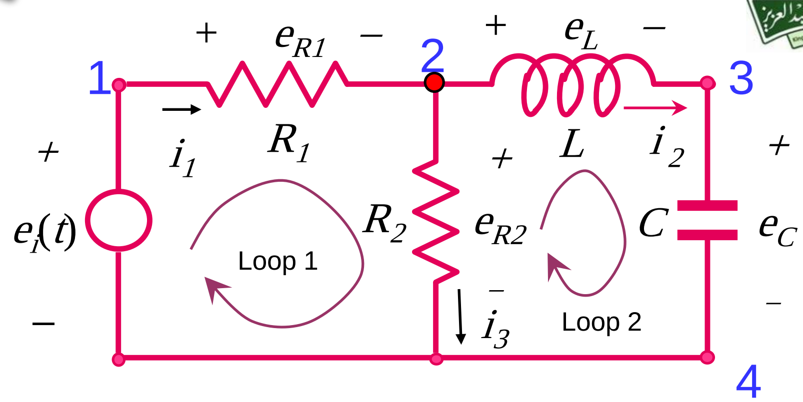
$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f$$

Mechanical rotational system

$$I_c\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = \tau$$

POP. Quiz

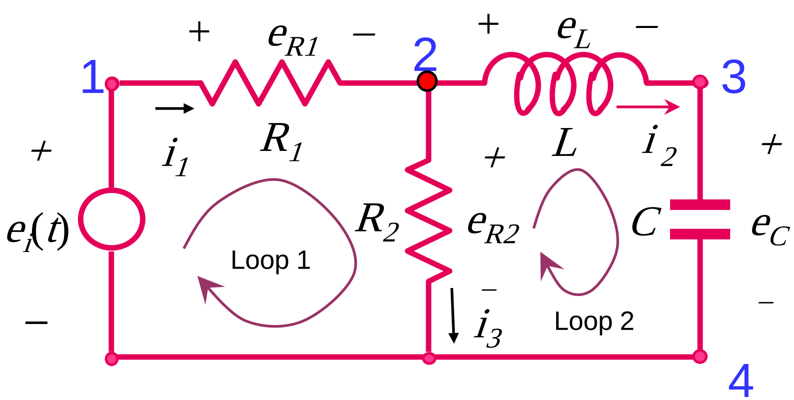
Obtain the I/O model for the following circuit. The input is the voltage $e_i(t)$ of the voltage source and the through current of the inductor is the output.



Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)\,d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Example

Obtain the I/O model for the following circuit. The input is the voltage $e_i(t)$ of the voltage source and the through current of the inductor is the output.



Elemental Equations:

$$e_{12} = e_{R1} = i_1 R_1, \quad e_{23} = e_L = L\left(\frac{d}{dt} i_2\right)$$

$$i_2 = C\left(\frac{d}{dt} e_{34}\right), e_{24} = -e_{42} = e_{R2} = i_3 R_2$$

$$e_{41} = -e_i(t)$$

Voltage Law

Loop 1: $e_{12} + e_{24} + e_{41} = 0$

Loop 2: $e_{23} + e_{34} + e_{42} = 0$

Current Law

Node 2: $i_1 - i_2 - i_3 = 0$

Unknown Variables

$$e_{12}, \quad e_{24}, \quad e_{41}, \quad e_{23}, \quad e_{34}, \quad e_{42},$$
$$i_1, \quad y = i_2, \quad i_3.$$

I/O Model

$$LC\frac{d^2 i_2}{dt^2} + \frac{R_1 R_2 C}{R_1 + R_2}\frac{di_2}{dt} + i_2 = \frac{R_2 C}{R_1 + R_2}\frac{de_i}{dt}$$

Example (cont.)

$$\begin{cases} i_2 = C\dfrac{de_{34}}{dt} \\ i_1R_1 + i_3R_2 - e_i = 0 \\ L\dfrac{di_2}{dt} + e_{34} - i_3R_2 = 0 \\ i_1 = i_2 + i_3 \end{cases} \Rightarrow \begin{cases} i_2 = C\dfrac{de_{34}}{dt} \\ i_2R_1 + i_3\left(R_1 + R_2\right) - e_i = 0 \\ LC\dfrac{d^2i_2}{dt^2} + C\dfrac{de_{34}}{dt} - \dfrac{di_3}{dt}CR_2 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} i_3 = \dfrac{-1}{R_1 + R_2}\left(i_2R_1 - e_i\right) \\ LC\dfrac{d^2i_2}{dt^2} + i_2 - \dfrac{di_3}{dt}CR_2 = 0 \end{cases} \Rightarrow LC\frac{d^2i_2}{dt^2} + i_2 + CR_2\frac{d}{dt}\underbrace{\frac{1}{R_1 + R_2}\left(i_2R_1 - e_i\right)}_{i_3} = 0$$

$$\boxed{LC\frac{d^2i_2}{dt^2} + \frac{R_1R_2C}{R_1 + R_2}\frac{di_2}{dt} + i_2 = \frac{R_2C}{R_1 + R_2}\frac{de_i}{dt}}$$

Basic relationship for electrical components

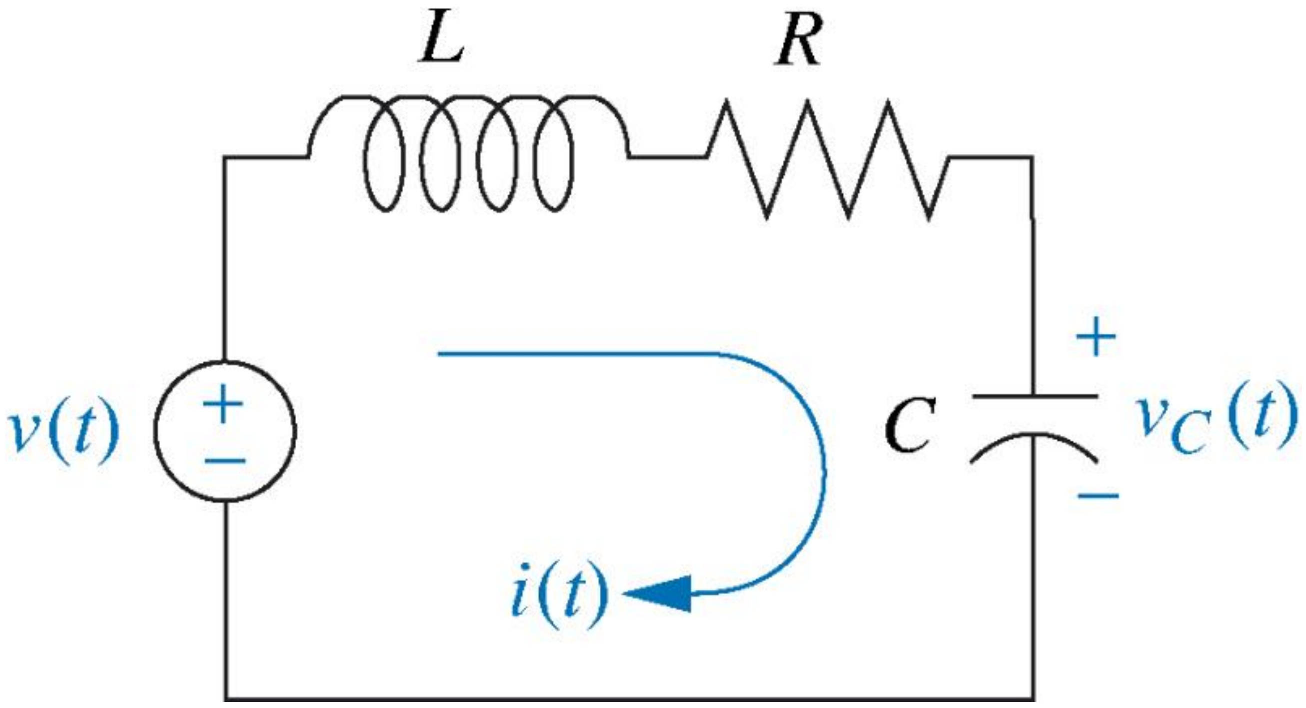
TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s)=V(s)/I(s)$	Admittance $Y(s)=I(s)/V(s)$
Capacitor	$v(t)=\frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t)=C\frac{dv(t)}{dt}$	$v(t)=\frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t)=Ri(t)$	$i(t)=\frac{1}{R}v(t)$	$v(t)=R\frac{dq(t)}{dt}$	R	$\frac{1}{R}=G$
Inductor	$v(t)=L\frac{di(t)}{dt}$	$i(t)=\frac{1}{L}\int_0^t v(\tau)\,d\tau$	$v(t)=L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

v(t)= V(volts), i(t)=current=A (Amp), q(t)=Q(coulombs), C=F(farads), R=Ω (ohms), L=H(henries)

Example 2.6 (Diff. Eqn.)

Find the transfer function V_c/V.



Solution

Summing the voltage around the loop gives:

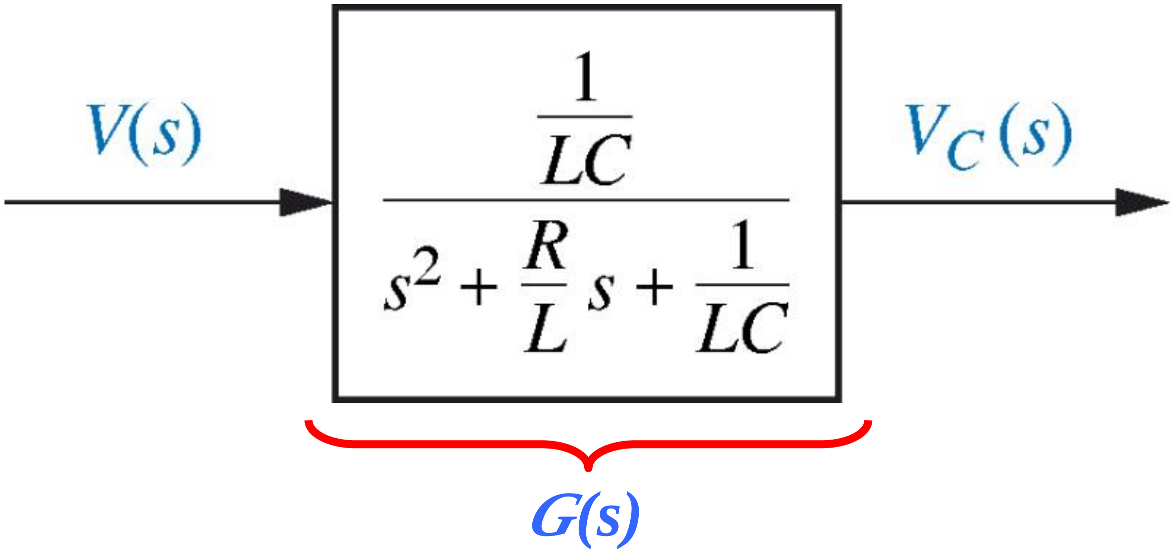
$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v$$

But, $i = \dfrac{dq}{dt}$ ⟹ $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v$$

as $q = Cv_c$ ⟹ $$LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c = v$$

Applying the Laplace Transform, gives

$$\left(LCs^2 + RCs + 1\right)V_c(s) = V(s)$$



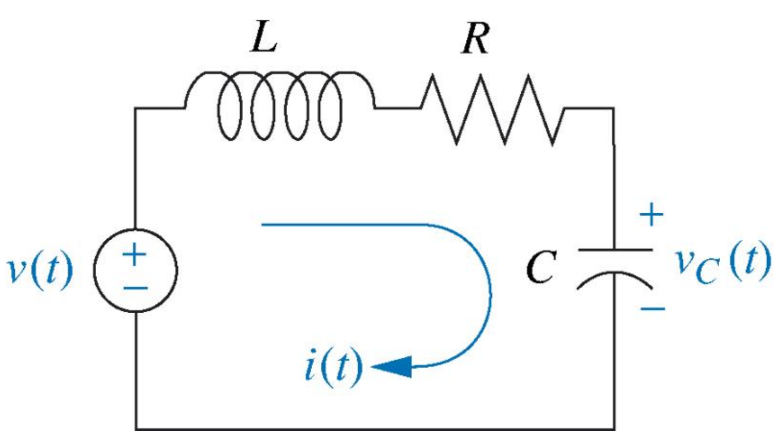
Component Transfer Function

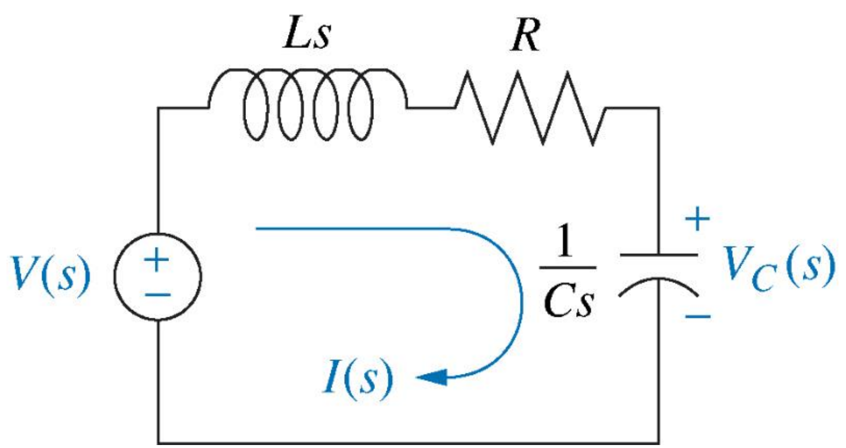
Capacitance (C) → $V(s) = \frac{1}{Cs} I(s)$

Resistance (R) → $V(s) = RI(s)$

Inductance (L) → $V(s) = LsI(s)$

→ $$\frac{V(s)}{I(s)} = Z(s)$$





Complex impedance $Z(s)$

Ratio of *E(s),* the Laplace transform (LP) of voltage across the terminal, to *I(s),* the Laplace transform (LP) of current through the element, under the assumption of zero initial conditions

$$Z(s) = \left. \frac{E(s)}{I(s)} \right|_{I.C.s=0}$$

$R, \ \frac{1}{Cs}$, and Ls, respectively.

Complex impedances of resistor, capacitor, inductor are

Series and Parallel Elements

- ***Parallel combinations***

  Same Voltage across elements
  $\Delta(Voltage_i) = \Delta(Voltage_j)$

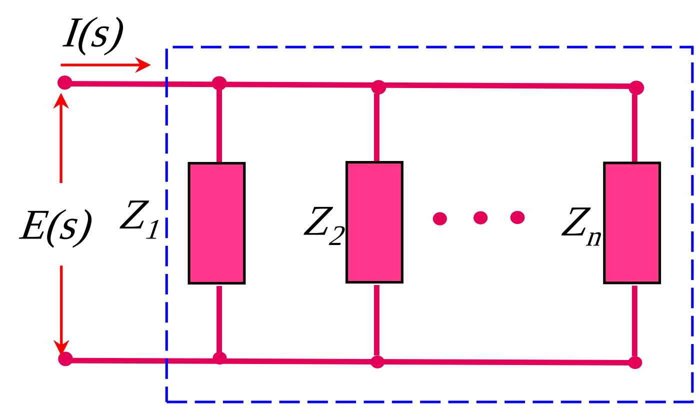
- ***Series combinations***

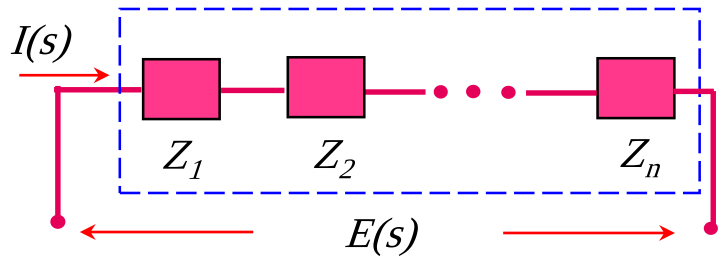
  Same Current through elements
  $(Current_i) = (Current_j)$

- ***Equivalent Complex Impedance***

$$\frac{1}{Z_{parallel}(s)} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)} + \cdots + \frac{1}{Z_n(s)}$$

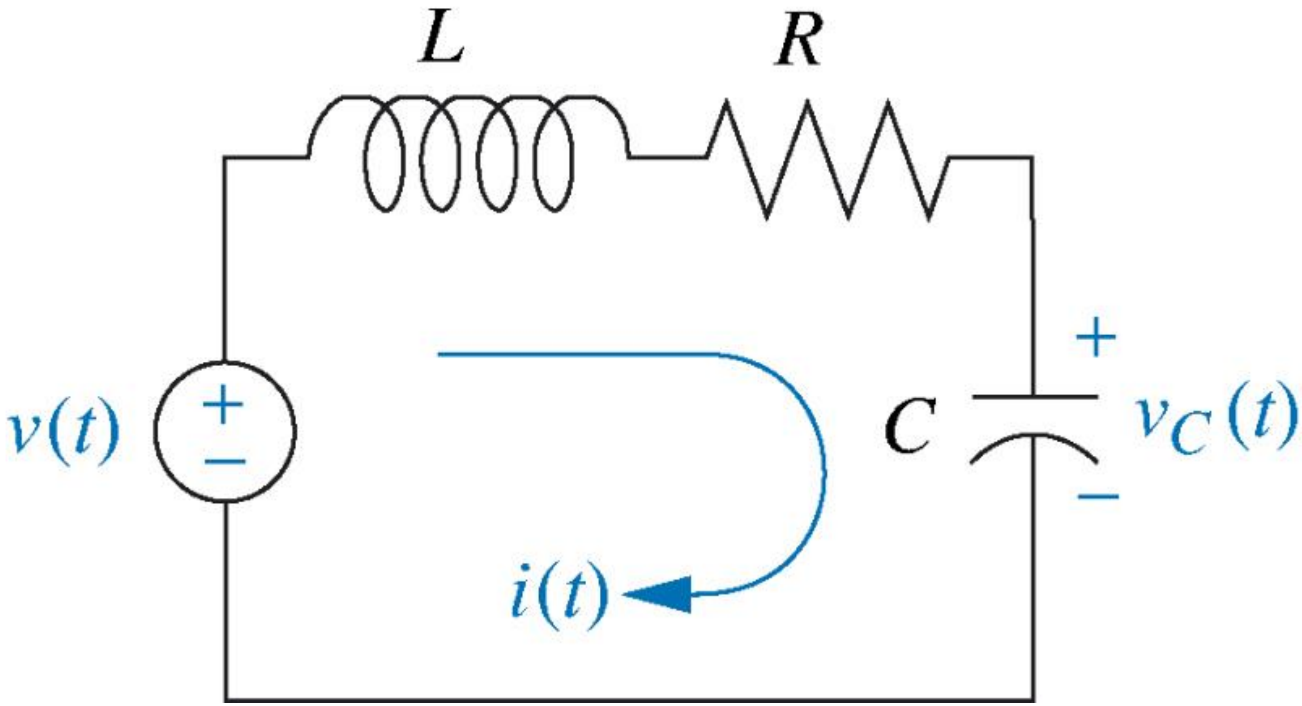
$$Z_{series}(s) = Z_1(s) + Z_2(s) + \cdots + Z_n(s)$$





Example (Transform Methods):

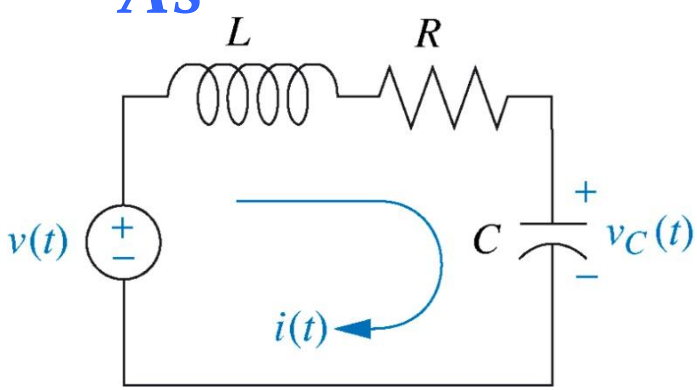
Find the transfer function V_c/V by mesh analysis & transform methods.

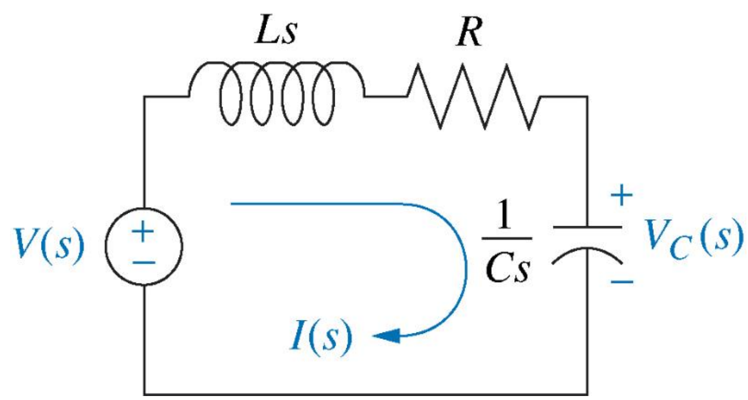


Circuit Analysis – Mesh Analysis

Solution

As



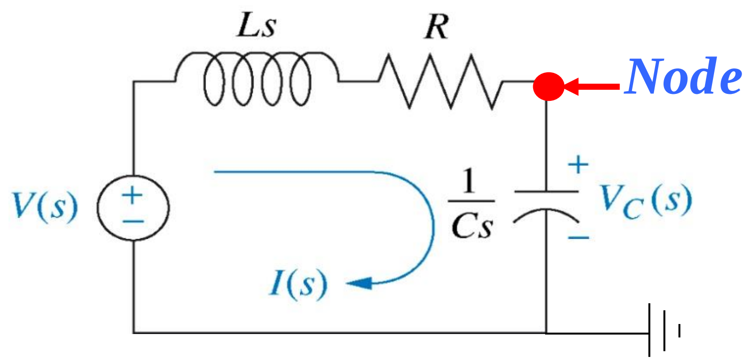


Then, as $Z(s) = Ls + R + \dfrac{1}{Cs}$ **,** $\dfrac{V(s)}{I(s)} = Z(s)$ **&** $\dfrac{V_c(s)}{I(s)} = \dfrac{1}{Cs}$

$$\frac{V_c(s)}{V(s)} = \frac{1}{\left(LCs^2 + RCs + 1\right)}$$

Example 2.8

Find the transfer function V_c/V by nodal analysis & transform methods.



Then, as $I(s)=\dfrac{V(s)}{Z(s)}$ ➡ *Kirchoff's Current Law at node*

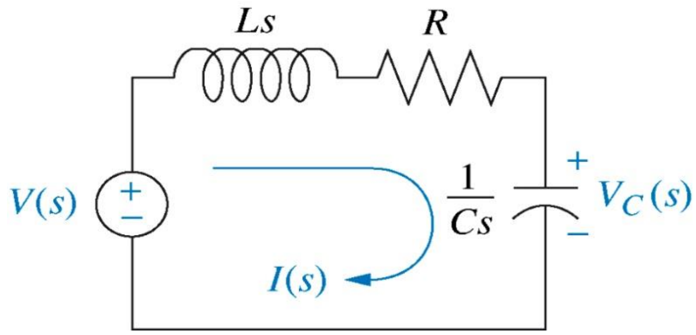
➡ $$\frac{V_c(s)}{1/Cs}+\frac{V_c(s)-V(s)}{Ls+R}=0$$

&

$$\frac{V_c(s)}{V(s)}=\frac{1}{\left(LCs^2+RCs+1\right)}$$

Example 2.9

Find the transfer function V_c/V by voltage division & transform methods.

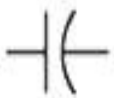
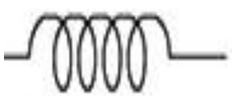


Voltage across capacitor is a portion of the input voltage which is proportional to the capacitor impedance to the sum of impedances.

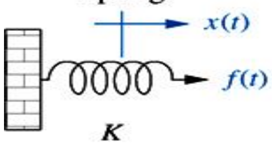
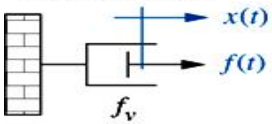
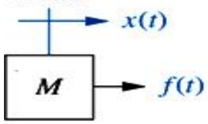
$$V_c(s) = \frac{1/Cs}{\left(Ls + R + 1/Cs\right)} V(s)$$

Basic Relationship for Electrical & Mechanical Components

Electrical Components

Component	Voltage-current	Current-voltage
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)\, d\tau$

Mechanical Components

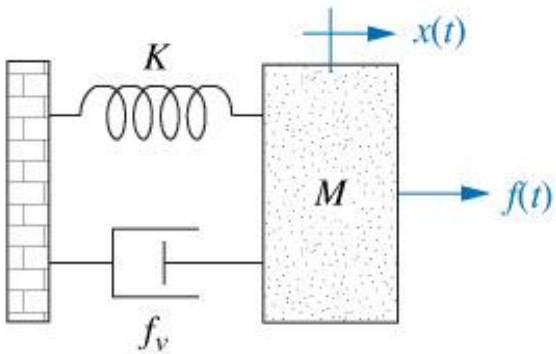
Component	Force-velocity
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$
Viscous damper 	$f(t) = f_v v(t)$
Mass 	$f(t) = M\frac{dv(t)}{dt}$

Two analogs

- *Voltage-current → Force-Velocity*
- *Current-voltage → Force-Velocity*

Electric Circuit Analogs

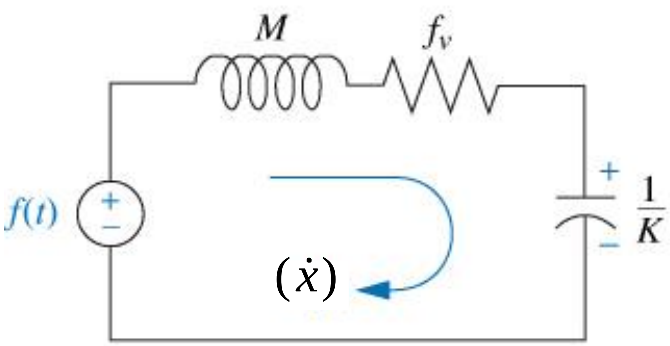
Series Analog



$$M\ddot{x} + f_v\dot{x} + Kx = f$$

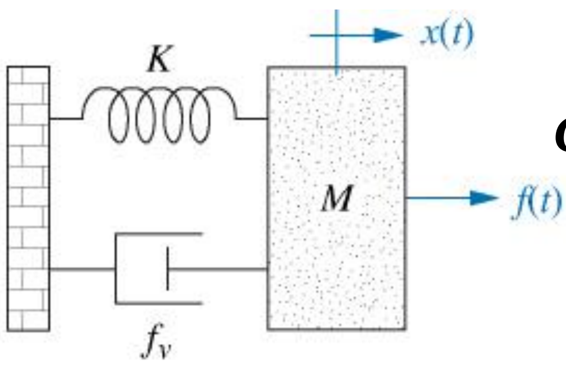
Voltage (v)-current (i) → Force (f)-Velocity $(\dot{x})$

$$M\frac{di}{dt} + f_v i + K\int i\,dt = v$$



mass = M	⟶	inductor	=	M henries
viscous damper = f_v	⟶	resistor	=	f_v ohms
spring = K	⟶	capacitor	=	$\frac{1}{K}$ farads

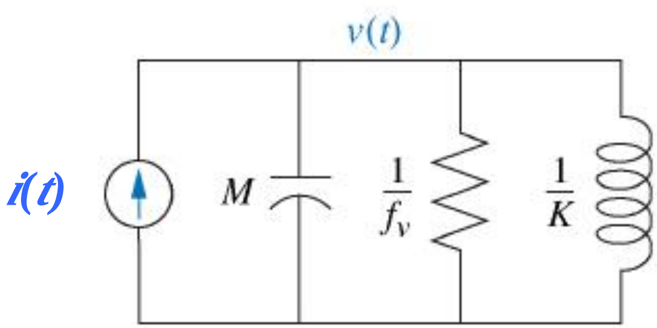
Parallel Analog



$$M\ddot{x} + f_v\dot{x} + Kx = f$$

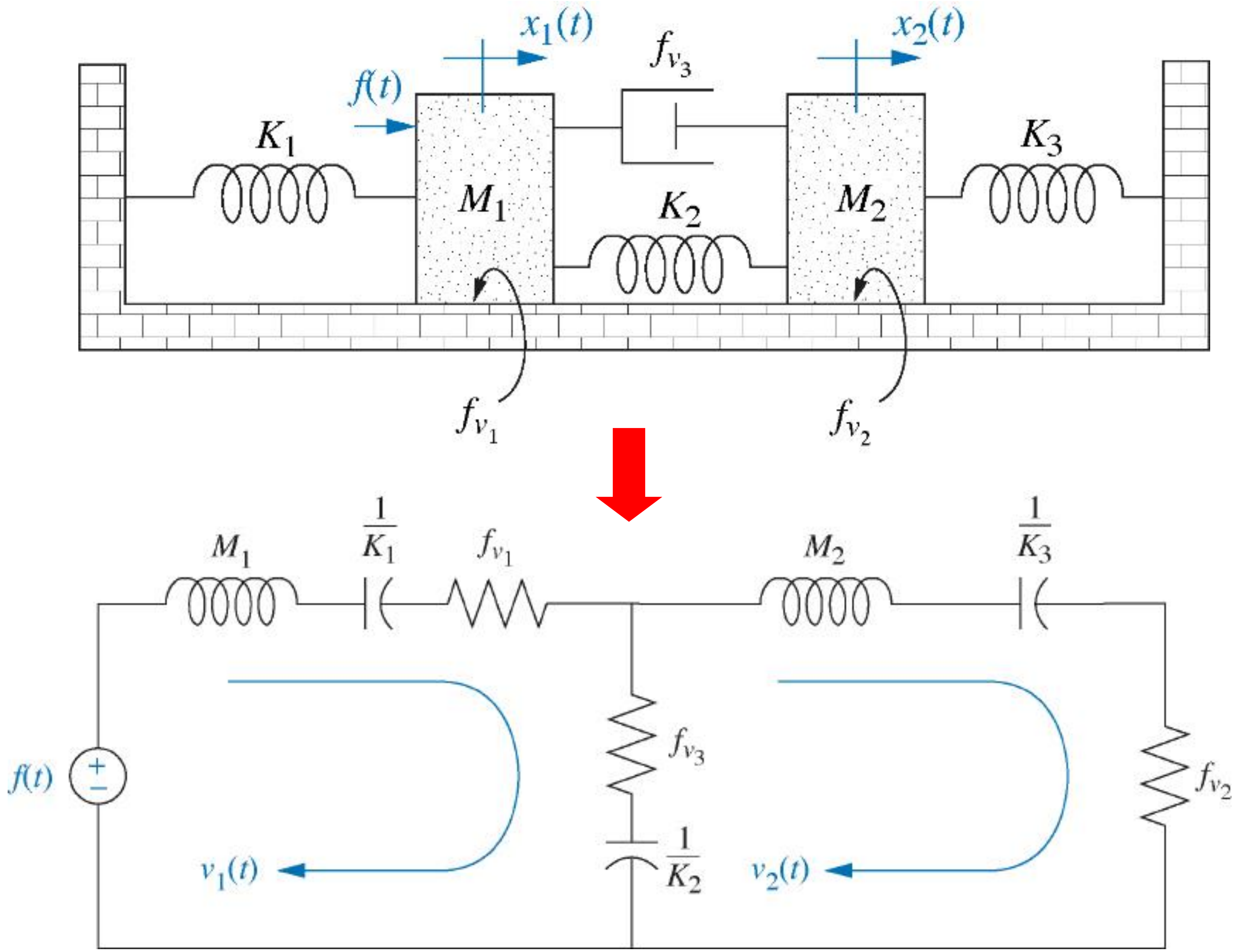
Current (i) -voltage (v)→ Force (f)-Velocity $(\dot{x})$

$$M\frac{dv}{dt} + f_v v + K\int v\,dt = i$$

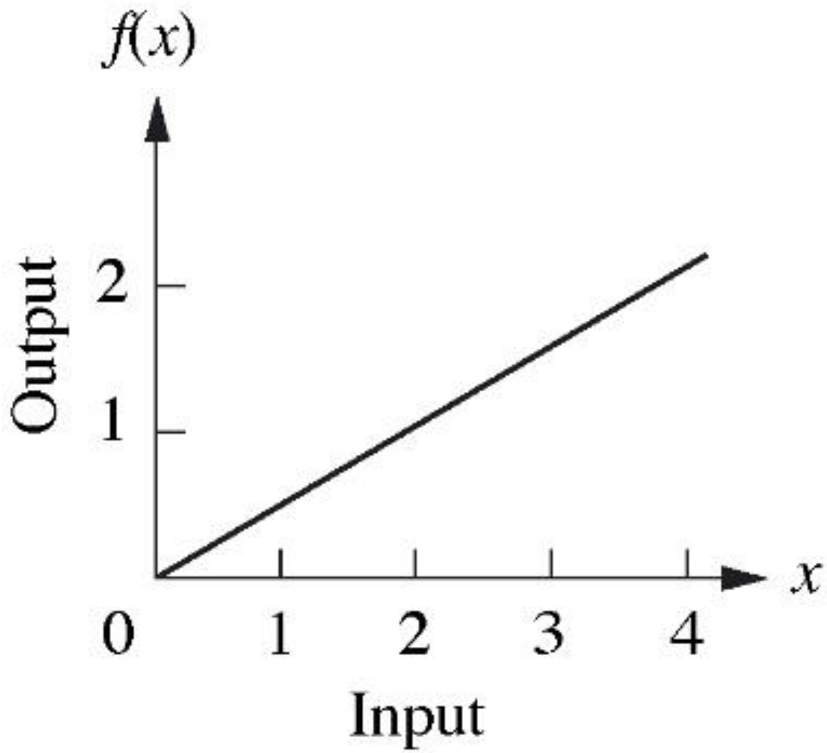


mass = M	⟶	capacitor	$= M$ farads
viscous damper = f_v	⟶	resistor	$= \frac{1}{f_v}$ ohms
spring = K	⟶	inductor	$= \frac{1}{K}$ henries

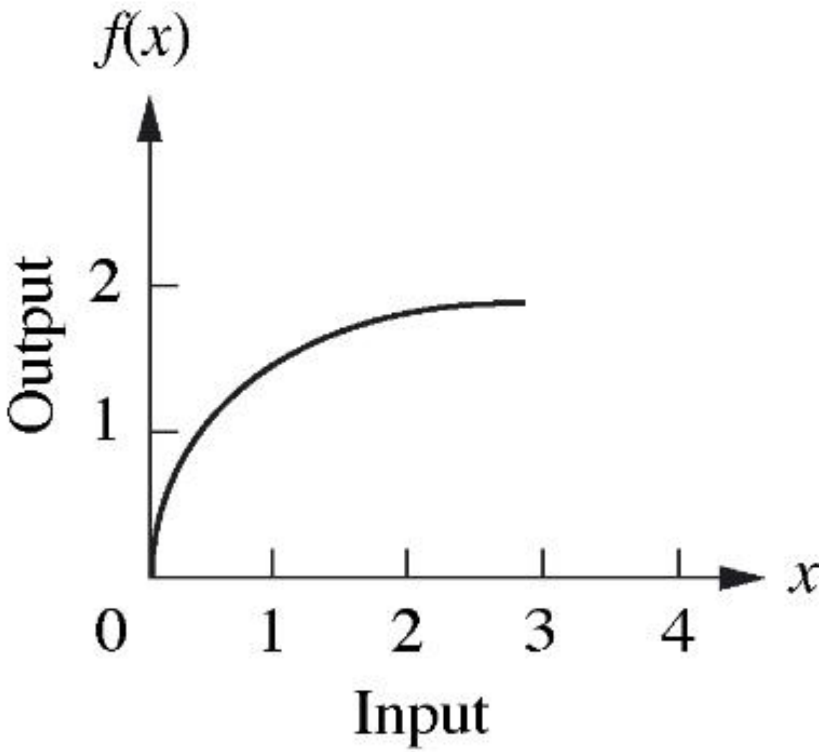
Electric Circuit Analogs



Nonlinearities

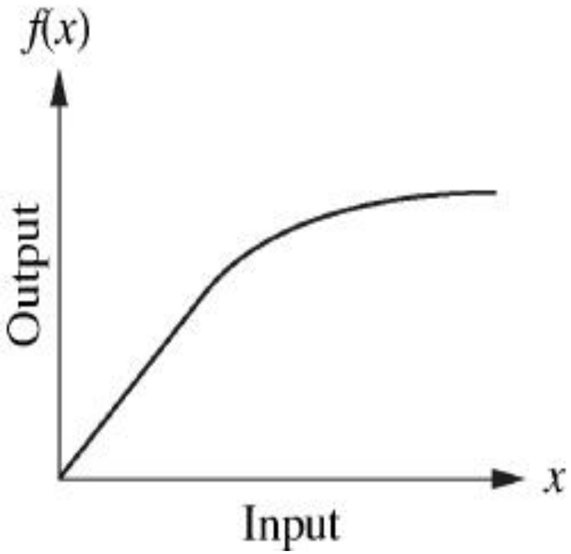


Linear System

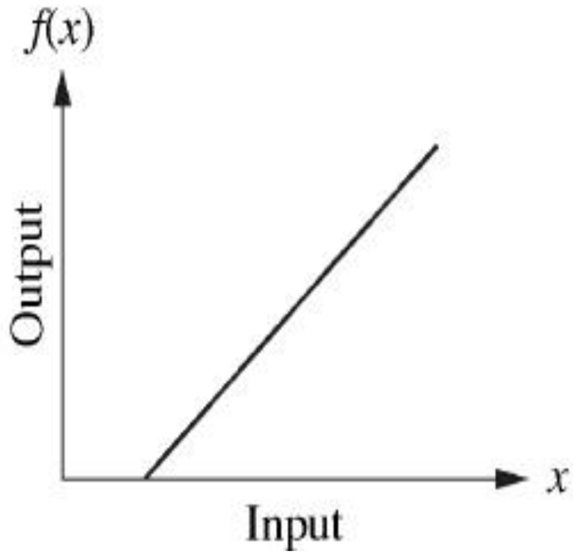


Nonlinear System

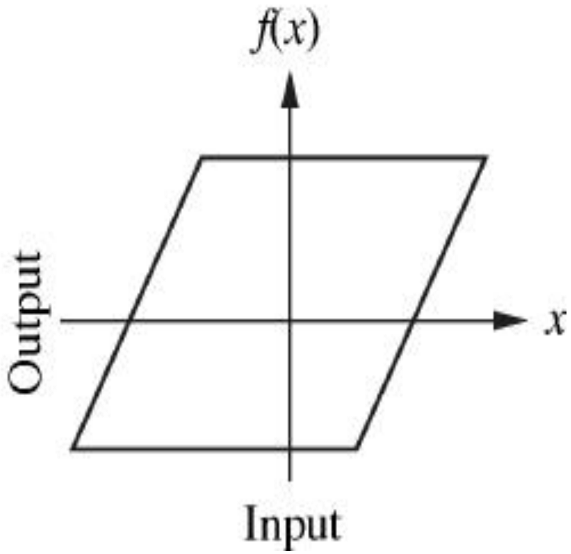
Nonlinearities



Amplifier Saturation



Motor Dead Zone



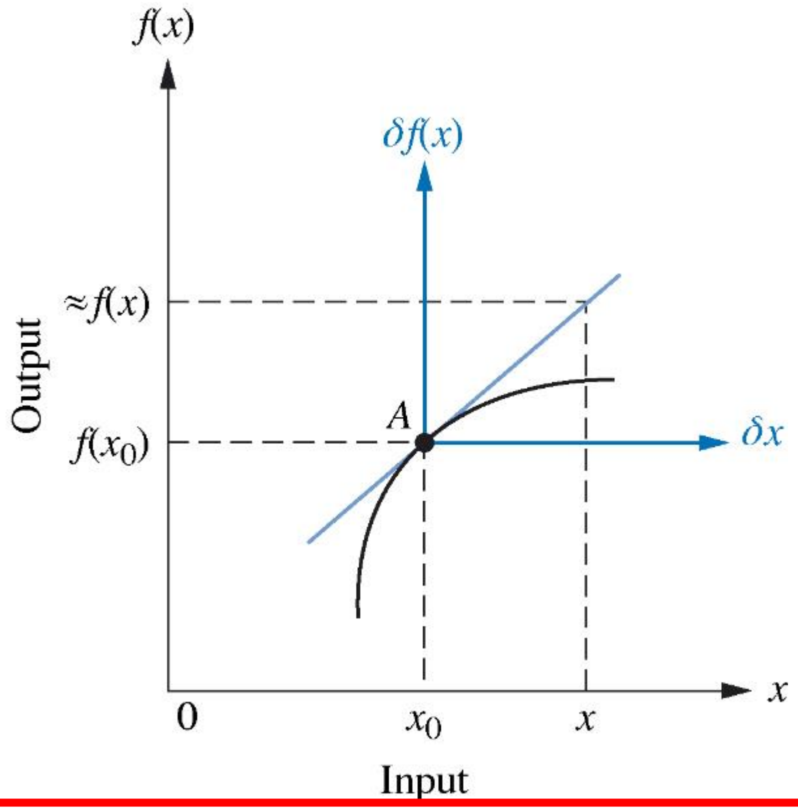
Backlash in Gears

Linearization of Nonlinearities

Using Taylor Series Expansion

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} \frac{(x-x_0)}{1!} + \left.\frac{d^2 f}{dx^2}\right|_{x=x_0} \frac{(x-x_0)^2}{2!} + \ldots.$$

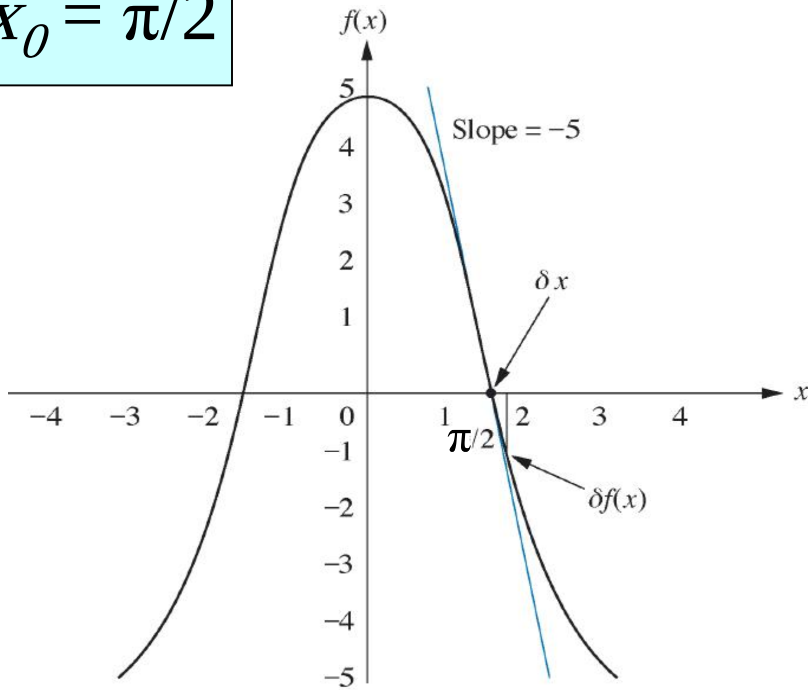
$$\cong f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} \frac{(x-x_0)}{1!}$$



$$\Rightarrow \quad f(x) - f(x_0) = \left.\frac{df}{dx}\right|_{x=x_0} (x-x_0) \quad \Rightarrow \quad \delta f(x) = \left.\frac{df}{dx}\right|_{x=x_0} \delta x$$

Linearization of Nonlinearities

Linearize $f(x) = 5\cos(x)$ about $x_0 = \pi/2$



As

$$f(x) - f(x_0) \cong \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$$

$$f(x) = 5\cos(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$$

$$= 5\cos(\pi/2) + \left(-5\sin x\right)_{x=\pi/2} (x - \pi/2)$$

$$= -5(x - \pi/2)$$

Linearization of Nonlinearities

Linearize the following system about $x_0 = \pi/4$

$$\ddot{x} + 2\dot{x} + \cos x = 0$$

Let $x = \delta x + \pi / 4$

$$\Rightarrow \quad \delta\ddot{x} + 2\delta\dot{x} + \cos(\delta x + \pi / 4)$$

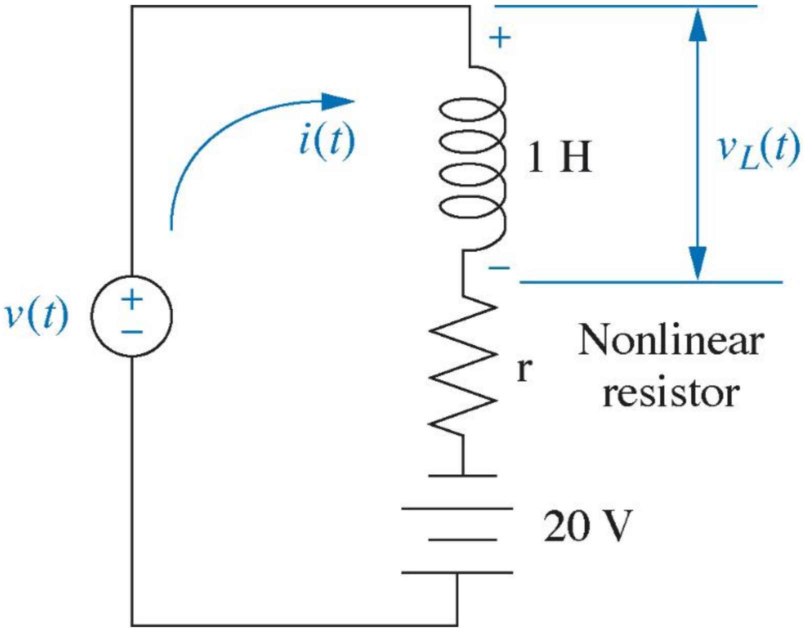
$$= \delta\ddot{x} + 2\delta\dot{x} + \cos(\pi / 4) - \sin(\pi / 4)\delta x = 0$$

or $\delta\ddot{x} + 2\delta\dot{x} - 0.707\delta x = -0.707$

$$\Rightarrow \quad \ddot{y} + 2\dot{y} - 0.707 y = -0.707 \qquad \textit{with} \quad y = \delta x$$

Linearization of Nonlinearities

Find the transfer function V_L/V



Solution

$$L\frac{di}{dt} + 10\ln(\tfrac{1}{2}i) - 20 = v$$

Equilibrium attained when

$$10\ln(\tfrac{1}{2}i_0) - 20 = 0$$

i_0=14.78 Amp.

Nonlinear Resistor

$$i = 2e^{0.1v} \Rightarrow v = 10\ln(\tfrac{1}{2}i)$$

Linearization of Nonlinearities

Linearize around equilibrium $i = \delta i + 14.78$

$$L\frac{d(i_0 + \delta i)}{dt} + 10\ln\tfrac{1}{2}(i_0 + \delta i) - 20$$

$$= L\frac{d\delta i}{dt} + 10\left(\ln\tfrac{1}{2} i_0 + \tfrac{1}{i_0}\delta i\right) - 20$$

$$= L\frac{d\delta i}{dt} + 0.677\delta i = v$$

Applying Laplace Transform

$$\delta i(s) = \frac{V(s)}{s + 0.677}$$

Linearization of Nonlinearities

But,

$$v_L = L\frac{di}{dt} = L\frac{d(i_0 + \delta i)}{dt} = L\frac{d\delta i}{dt}$$

Applying Laplace Transform

$$\Rightarrow \quad \delta i(s) = V_L / Ls$$

Combining with,

$$\delta i(s) = \frac{V(s)}{s + 0.677}$$

$$\Rightarrow \quad \frac{V_L}{V} = \frac{Ls}{s + 0.677}$$

END