MENG366

Transient and Steady State Response of First, Second, and Higher Order Systems

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Forced Responses of LTI Systems

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_m u^{(m)} + \cdots + b_1\dot{u} + b_0 u$$

- **Superposition Principle**

$u_1(t)$ $\quad$ $y_1(t)$

$u_2(t)$ $\quad$ $y_2(t)$

$u(t)=k_1 u_1(t) + k_2 u_2(t)$ $\quad$ $y(t)=k_1 y_1(t) + k_2 y_2(t)$

$\text{complicated input}=\sum \text{simple inputs}$ $\quad$ $\sum \begin{matrix}\text{forced responses of} \\ \text{simple inputs}\end{matrix} = \text{forced response of complicated input}$

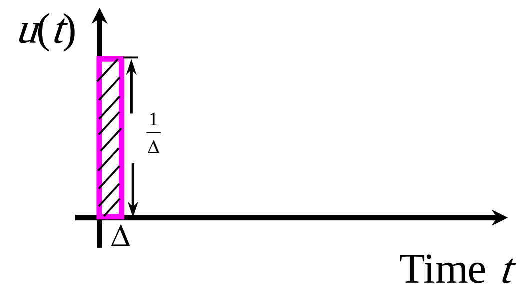
The forced response of a linear system to a complicated input can be obtained by studying how the system responds to simple inputs, such as unit impulse input, unit step input, and sinusoidal inputs with different input frequencies.

Typical Forced Responses

- **Unit Impulse Response**
  - Forced response to unit impulse input

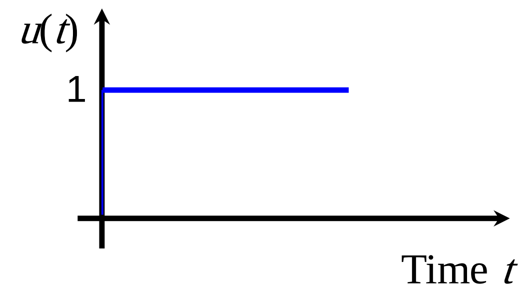
$$u(t)=\delta(t)=\begin{cases}+\infty, & t=0\\ 0, & t\neq 0\end{cases},\quad \int_{-\infty}^{+\infty}\delta(t)\,dt=1 \quad Y(s)=G(s)U(s)=G(s)$$

If system is stable, *SS* is zero.



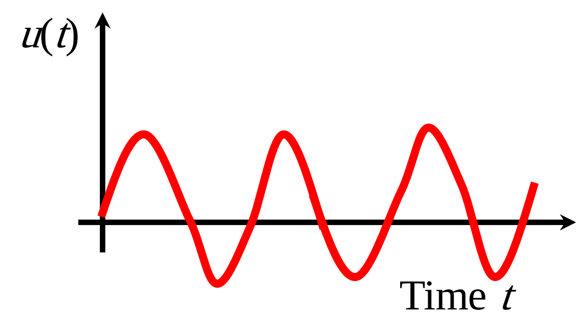
- **Unit Step Response**
  - Forced response to unit step input (u (t) = 1)

$$Y(s)=G(s)U(s)=\underbrace{\frac{1}{s}}_{U(s)}G(s) \qquad y(\infty)=\lim_{s\to 0} sY(s)=\lim_{s\to 0} s\frac{1}{s}G(s)=G(0)$$



- **Sinusoidal Response**
  - Forced response to sinusoidal inputs at different input frequencies
  - The *steady state response* of sinusoidal response is call the *Frequency Response*.

$$Y(s)=G(s)U(s)=G(s)\underbrace{\frac{\omega}{s^2+\omega^2}}_{U(s)} \qquad sY(s)=\frac{s\omega}{s^2+\omega^2}G(s)$$



Forced Response of 1st Order Systems

- **Standard Form of *Stable* 1st Order System**

$$\dot{y} + ay = bu \quad \Rightarrow \quad \boxed{\tau\dot{y} + y = Ku}$$

$$\frac{1}{a}\dot{y} + y = \frac{b}{a}u$$

where

τ : Time Constant $\qquad \tau = \dfrac{1}{a}$

K : Static (Steady State, DC) Gain $\qquad K = \dfrac{b}{a}$

- **TF and Poles/Zeros**

Stable system

$$G(s) = \frac{K}{\tau s + 1}$$

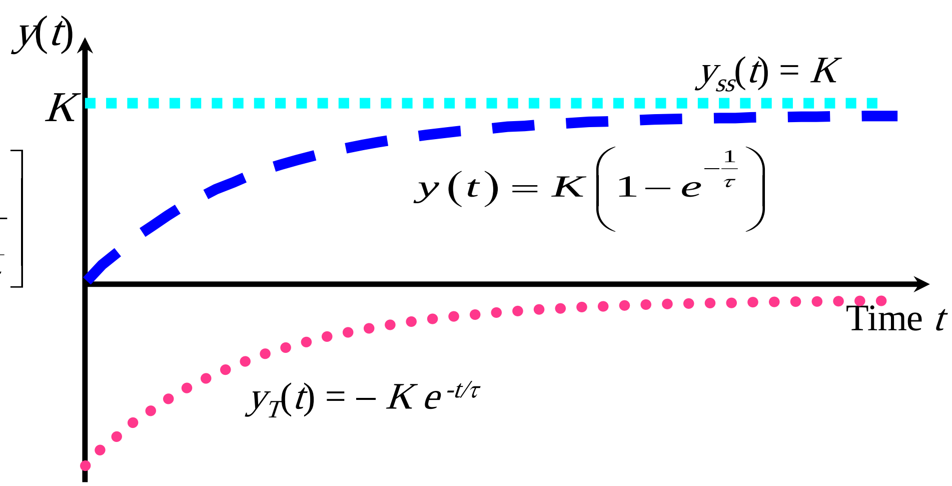
pole: $p = -\dfrac{1}{\tau} < 0$

zero: No zero

- **Unit Step Response**
  - (u=1 and zero ICs)

$$Y(s) = G(s)U(s) = \frac{K}{(\tau s+1)s} = K\left[\frac{1}{s} - \frac{\tau}{\tau s+1}\right] = K\left[\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right]$$

$$y(t) = K\left(1 - e^{-\frac{1}{\tau}}\right)$$



Normalized Unit Step Response

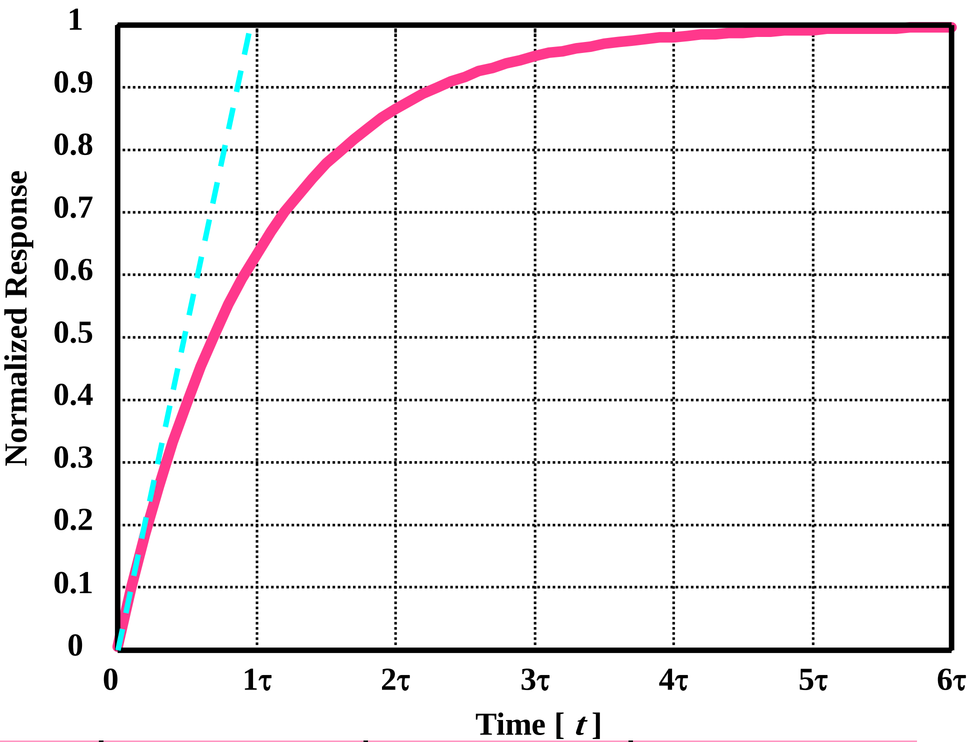
Normalized Unit Step Response (u = 1 & zero ICs)

$$\tau \dot{y} + y = Ku$$

$$\Rightarrow \quad y(t) = K(1 - e^{-t/\tau})$$

Normalized
(such that as $t \to \infty$, $y_n \to 1$):

$$\Rightarrow \quad y_n(t) = \frac{y(t)}{K} = 1 - e^{-t/\tau}$$



Time t	τ	2τ	3τ	4τ	5τ
$(1 - e^{-t/\tau})$	0.6321	0.8647	0.9502	0.9817	0.9933

Unit Step Response of Stable 1st Order System

Effect of Time Constant τ :

$$\tau\dot{y} + y = Ku$$

$$\Rightarrow \; y(t) = K(1 - e^{-t/\tau})$$

Normalized:

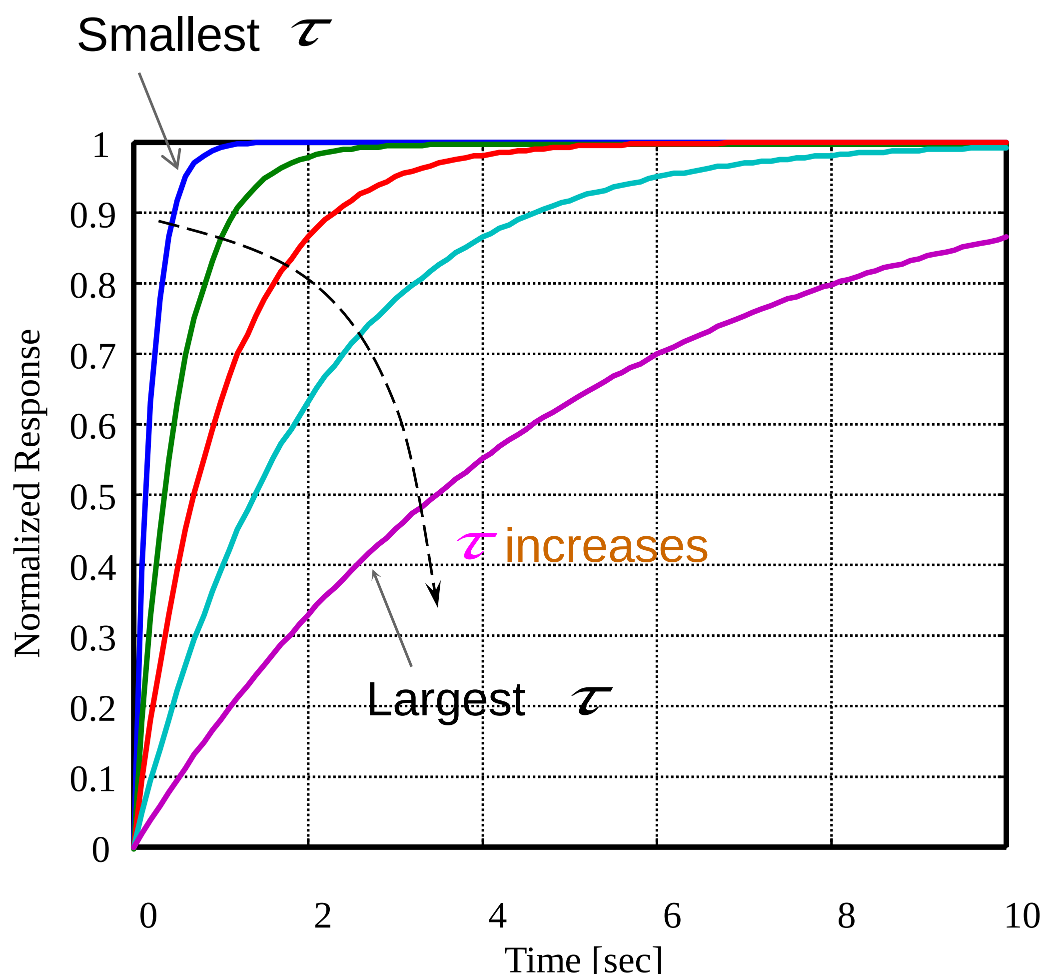
$$\Rightarrow \; y_n(t) = \frac{y(t)}{K} = (1 - e^{-t/\tau})$$

Initial Slope:

$$\Rightarrow \; \frac{d}{dt}y_n(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}$$

$$\Rightarrow \; \frac{d}{dt}y_n(0) = \frac{1}{\tau}$$

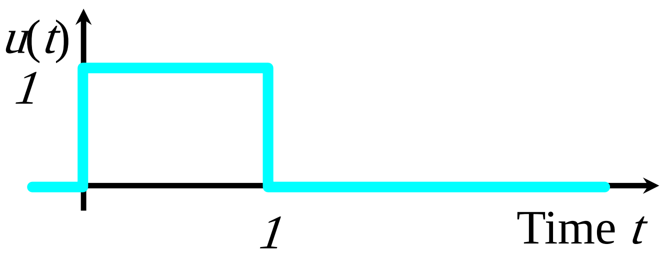
Q: What is your conclusion ?



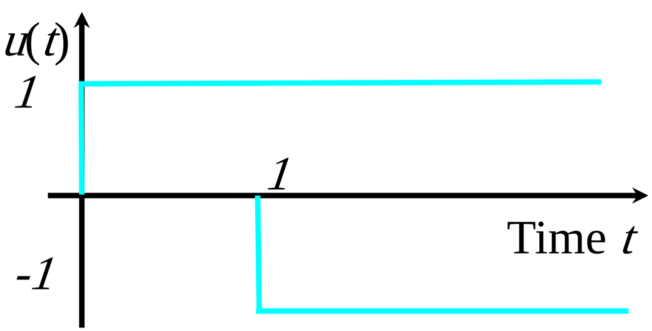
The smaller τ is,
the steeper the initial slope is, and
the faster the response approaches the steady state.

Forced Responses of Stable 1st Order System

Q: *How would you calculate the forced response of a 1st order system to a <u>unit pulse</u> (not unit impulse)?*



Q (Hint: superposition principle ?!)



$$u(t) = u_s(t) - u_s(t-1)$$

$$y(t) = y_s(t) - y_s(t-1)$$

Q: *How would you calculate the unit impulse response of a 1st order system?*

$$\delta(t) = \frac{d}{dt} u_s(t)$$

$$y(t) = \frac{d}{dt} y_s(t)$$

Q: *How would you calculate the sinusoidal response of a 1st order system?*

Standard Form of 2nd Order Systems

- **I/O Model**

$$\ddot{y} + a_1\dot{y} + a_0 y = b_1\dot{u} + b_0 u$$

- **TF and Pole/Zeros**

$$G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \qquad \text{pole: } p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^{\,2} - 4a_0}}{2}$$

$$\omega_n = \sqrt{a_0} \qquad \text{zero: } z = -\frac{b_0}{b_1}$$

- **Stability Condition**

$$a_1 > 0, \quad a_0 > 0$$

- **Standard Form of *Stable* 2nd Order Systems without Zeros**

$$\ddot{y} + a_1\dot{y} + a_0 y = b_0 u \quad \Rightarrow \quad \ddot{y} + \underbrace{2\zeta\omega_n}_{a_1}\dot{y} + \underbrace{\omega_n^{\,2}}_{a_0} y = \underbrace{K\omega_n^{\,2}}_{b_0} u$$

where

ω_n : **Natural Frequency [rad/s]**

ζ : **Damping Ratio** $\quad \varsigma = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$

K : **Static (Steady State, DC) Gain** $\quad K = \frac{b_0}{\omega_n^{\,2}} = \frac{b_0}{a_0}$

Poles of Stable 2nd Order Systems

- **Stable 2nd Order Systems without Zeros**

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n{}^2 y \;=\; K\omega_n{}^2 u$$

$$G(s) = \frac{K\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} = \frac{K}{\frac{1}{\omega_n{}^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

- **Pole Locations**

$$s^2 + 2\zeta\omega_n s + \omega_n{}^2 = 0$$

$$\Rightarrow \quad p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{(\zeta^2 - 1)}$$

– **Over-damped ($\zeta > 1$)**

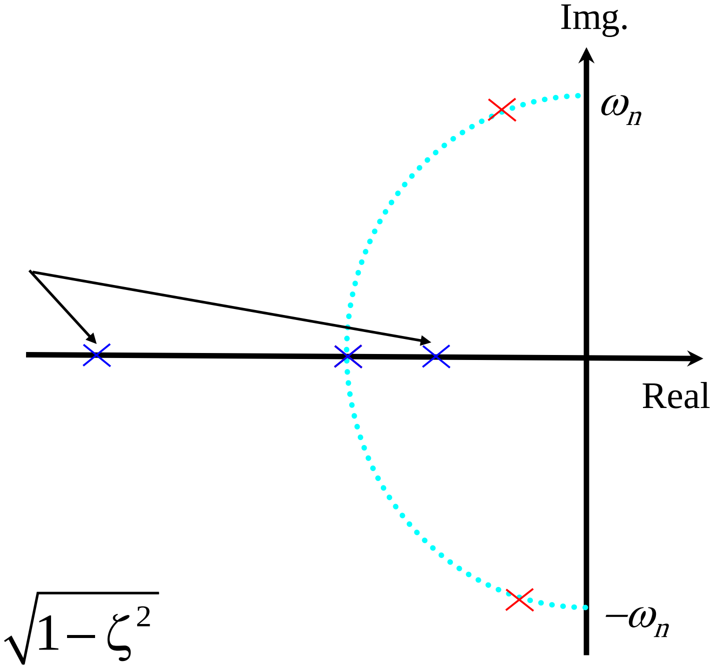
Two distinct real poles

– **Critically damped ($\zeta = 1$)**

Two identical real poles at $p_{1,2} = -\zeta\omega_n$

– **Under damped ($\zeta < 1$)**

Two complex poles at $p_{1,2} = \underbrace{-\zeta\omega_n}_{\sigma} \pm j\,\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$



Under-damped 2nd Order System

- **Unit Step Response** (u=1 and zero ICs)

$$Y(s)=G(s)\frac{1}{s}=\frac{K\omega_n^{\ 2}}{s\left(s^2+2\zeta\omega_n s+\omega_n^{\ 2}\right)}=\frac{K\omega_n^{\ 2}}{s(s-p_1)(s-p_2)},\quad p_{1,2}=-\sigma\pm j\omega_d$$

$$=\frac{K}{s}+\frac{A_1}{s+\sigma-j\omega_d}+\frac{\bar{A}_1}{s+\sigma+j\omega_d},\qquad A_1=-\frac{K}{2}\left(1-j\frac{\sigma}{\omega_d}\right)$$

$$\Rightarrow\quad y(t)=K+\underbrace{A_1 e^{(-\sigma+j\omega_d)t}+\bar{A}_1 e^{(-\sigma-j\omega_d)t}}_{2\,\mathrm{Re}\left\{A_1 e^{(-\sigma+j\omega_d)t}\right\}}$$

$$=K\left[1-e^{-\sigma t}\left(\cos\left(\omega_d t\right)+\frac{\varsigma\omega_n}{\omega_d}\sin\left(\omega_d t\right)\right)\right]$$

$$=K-\frac{K}{\sqrt{1-\varsigma^2}}e^{-\sigma t}\sin\left(\omega_d t+\varphi\right),\qquad \varphi=\tan^{-1}\left(\frac{\sqrt{1-\varsigma^2}}{\varsigma}\right)$$

Under-damped 2nd Order System

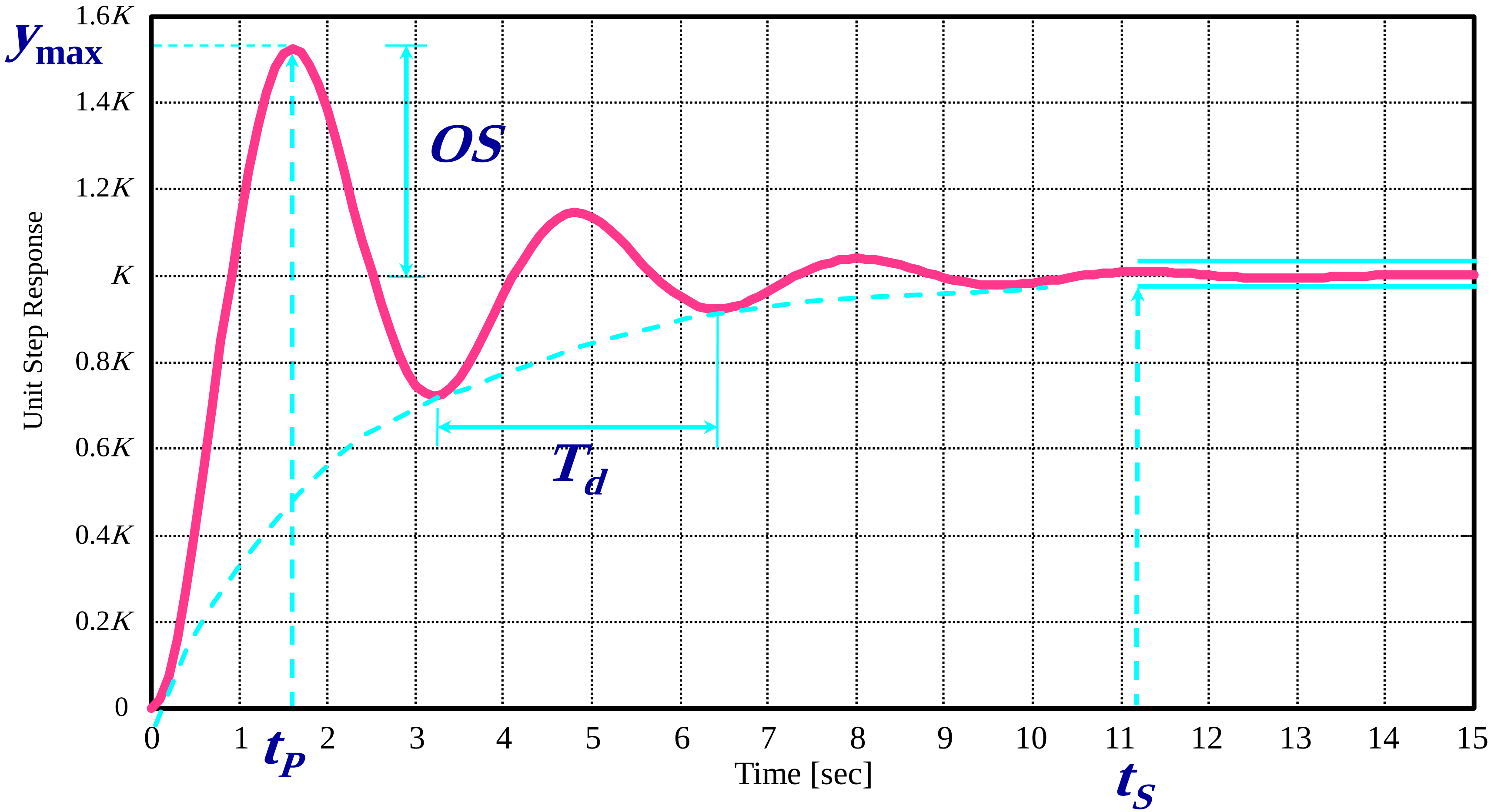
- **Unit Step Response** (u=1 and zero ICs)

$$Y(s) = G(s)\frac{1}{s} = K\left[\frac{\omega_n^{\ 2}}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}\right)}\right]$$

<u>From Laplace Table:</u>

$$y(t) = K\left[1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\sigma t}\sin\left(\omega_d t + \varphi\right)\right], \qquad \varphi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Unit Step Response of 2nd Order Systems



Unit Step Response of 2nd Order System

- **Peak Time (t_P)**

Time when output $y(t)$ reaches its maximum value y_{MAX}.

$$y(t) = K\left[1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\varsigma\omega_n}{\omega_d}\sin(\omega_d t)\right)\right]$$

$$\Rightarrow$$

$$\frac{dy}{dt} = Ke^{-\varsigma\omega_n}\frac{\omega_n}{\sqrt{1-\varsigma^2}}\sin(\omega_d t)$$

Find t_P such that $\dot{y}(t_P) = 0$

$$\omega_d t_P = \pi$$

$$\Rightarrow$$

$$t_P = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$T_d = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}$$

- **Percent Overshoot (%OS)**

At peak time t_P the maximum output

$$y_{MAX} = y(t_P) = K\left(1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}}\right)$$

The overshoot (OS) is:

$$OS = y_{MAX} - y_{SS} = Ke^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}}$$

The percent overshoot is:

$$\%OS = \left(\frac{OS}{y_{SS} - y(0)}\right)100\%$$

$$= e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}} \times 100\%$$

Unit Step Response of 2nd Order System

- **Settling Time (t_s)**

Time required for the response to be within a specific percent of the final (steady-state) value.

Some typical specifications for settling time are: 5%, 2% and 1%.

Look at the envelope of the response:

***x*% band settling time:**

$$t_s = -\tau \ln\left(x/100 \right) \approx$$

$$\tau = 1/\sigma$$

%	1%	2%	5%
t_S	4.6τ	3.9τ	3τ

Q: Which parameters of a 2nd order system affect the peak time?

$$t_P = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Q: Which parameters of a 2nd order system affect the % OS?

$$\%OS = e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}} \times 100\%$$

Q: Which parameters of a 2nd order system affect the settling time?

Q: Can you obtain the formula for a 3% settling time?

Unit Step Response of 2nd Order System

- **Settling Time (t_s)**

  Time required for the response to be within a specific percent of the final (steady-state) value.

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$$t_P = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Damping ration and natural frequency

Q: Which parameters of a 2nd order system affect the % OS?

Damping ratio $\%OS = e^{-\frac{\pi\varsigma}{\sqrt{1-\varsigma^2}}} \times 100\%$

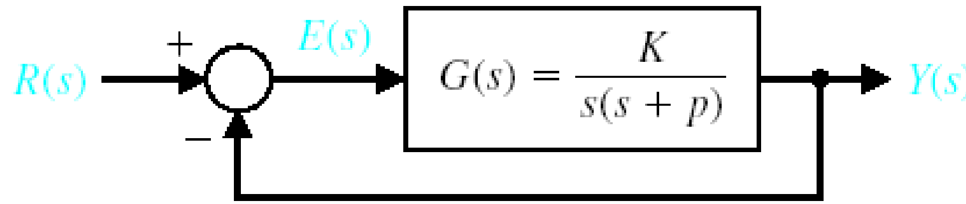
Q: Which parameters of a 2nd order system affect the settling time?

Damping ratio and natural frequency

Q: Can you obtain the formula for a 3% settling time?

Performance of a Second-Order System

$$Y(s) = \frac{K}{s^2 + p \cdot s + K} \cdot R(s)$$



$$Y(s) = \frac{\omega_n^2}{s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^2}$$

with a unity step input

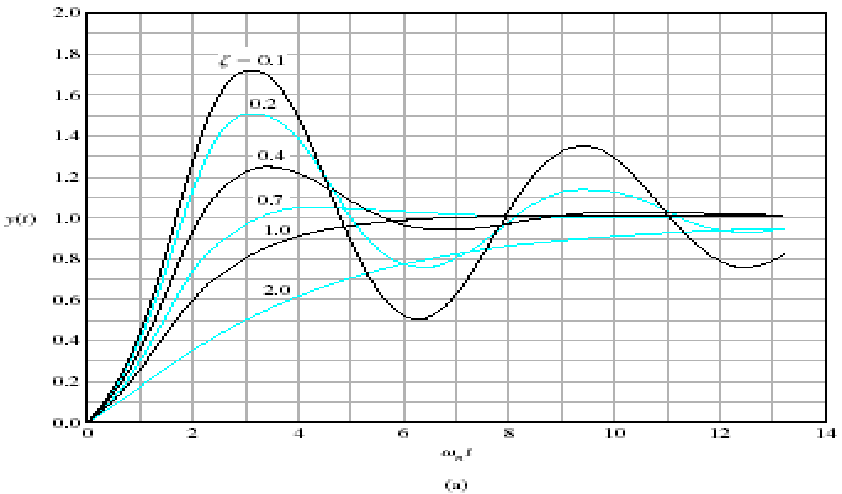
$$\beta = \sqrt{1 - \zeta^2}$$

$$\theta = \cos^{-1}(\zeta)$$

$$Y(s) = \frac{\omega_n^2}{\left(s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^2\right) \cdot s}$$

$$y(t) = 1 - \frac{1}{\beta} \cdot e^{-\zeta \cdot \omega_n \cdot t} \cdot \sin\left(\omega_n \cdot \beta \cdot t + \theta\right)$$

Performance of a Second-Order System



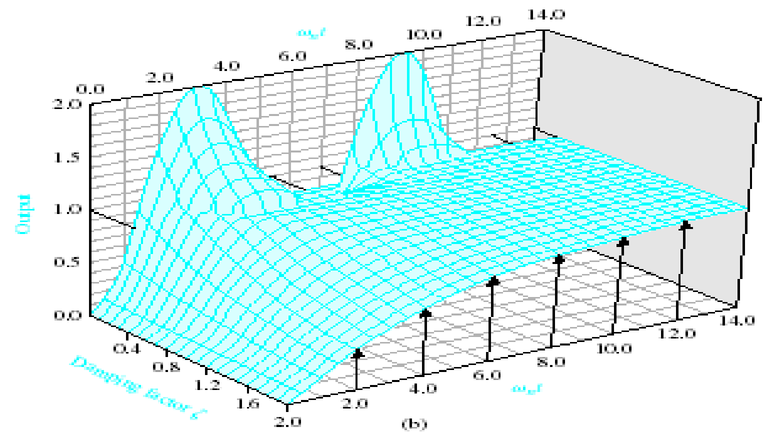
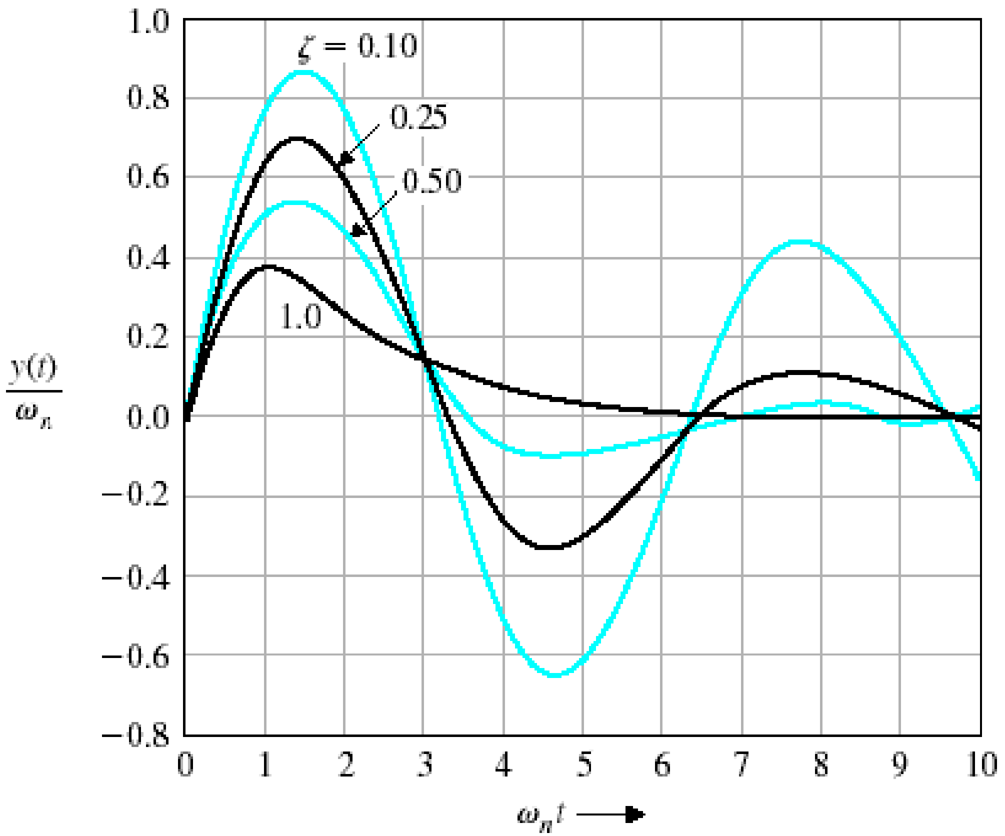


FIGURE 5.5

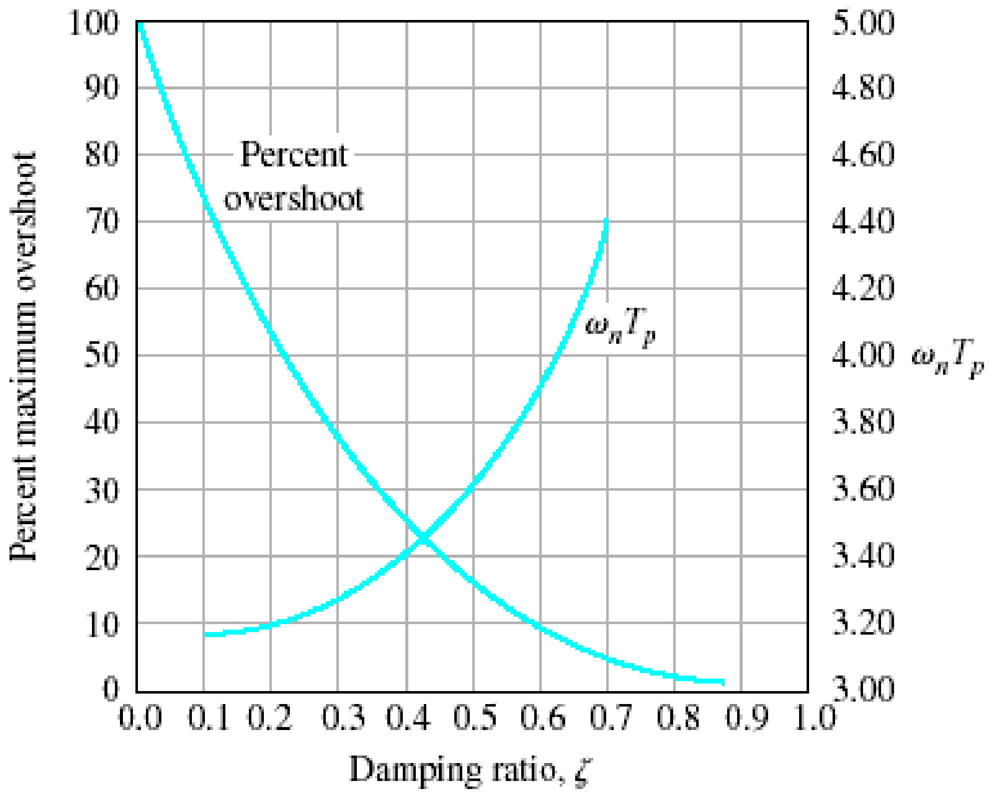
(a) Transient response of a second-order system (Eq. 5.9) for a step input. (b) The transient response of a second-order system (Eq. 5.9) for a step input as a function of ζ and $\omega_n t$. (Courtesy of Professor R. Jacquot, University of Wyoming.)

Performance of a Second-Order System



Response of a second-order system for an impulse function input.

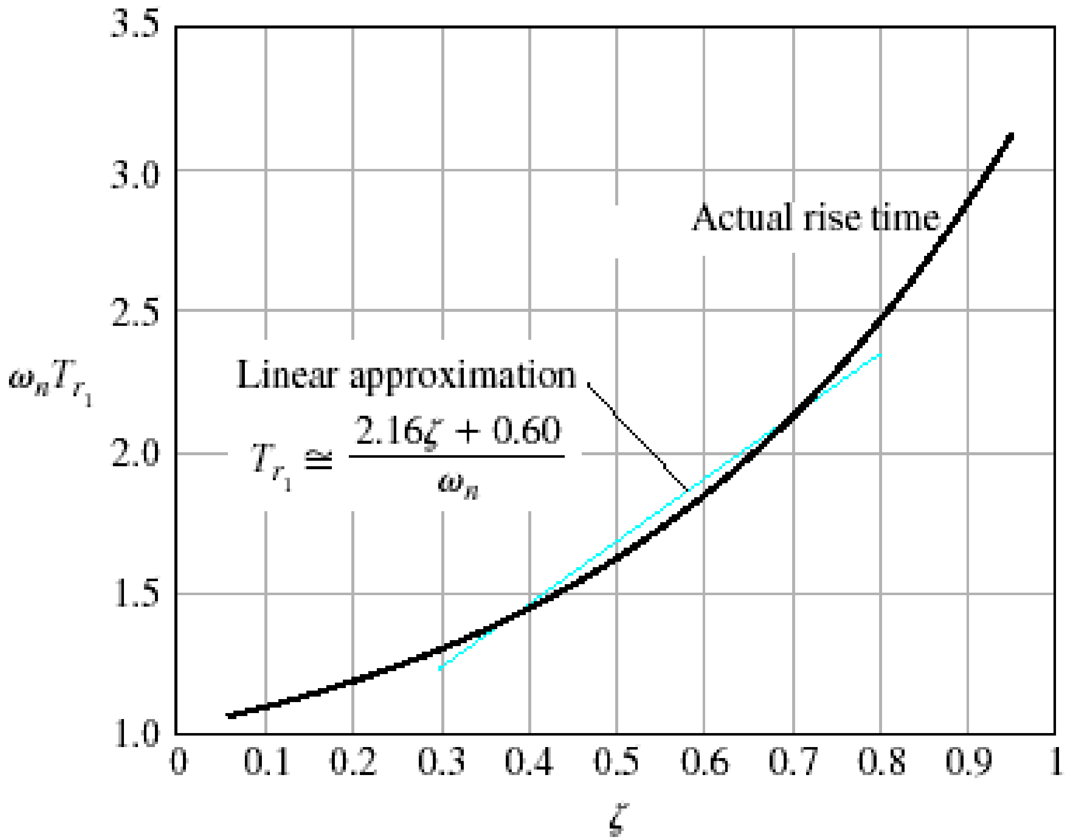
Performance of a Second-Order System



Percent overshoot and normalized peak time versus damping ratio ζ for a second-order system (Eq. 5.8).

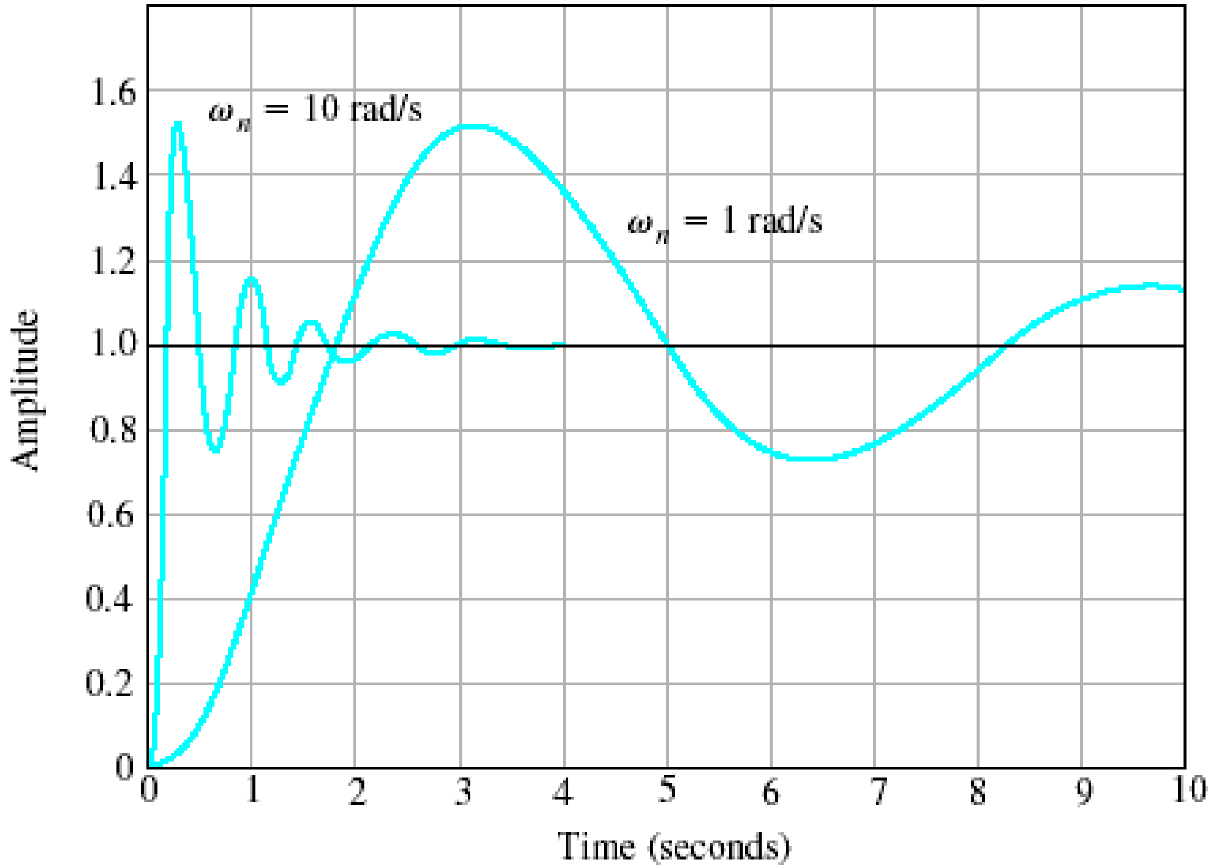
Naturally these two performance measures are in opposition and a compromise must be made.

Performance of a Second-Order System



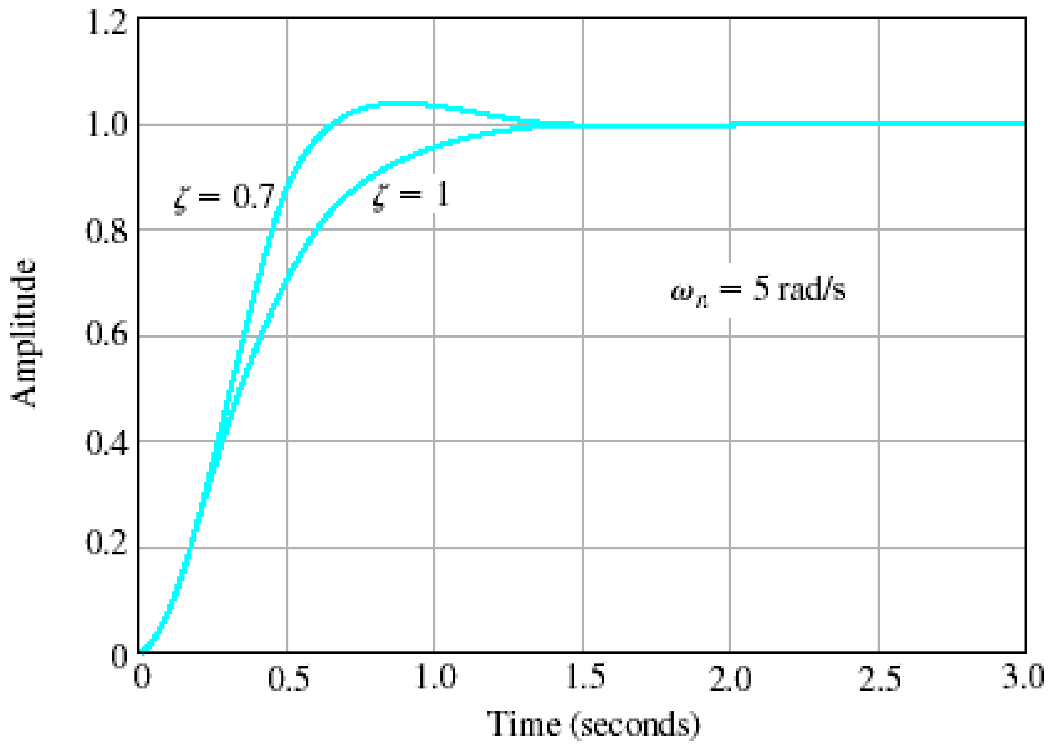
Normalized rise time T_{r1} versus ζ for a second-order system.

Performance of a Second-Order System



The step response for $\zeta = 0.2$ for $\omega_n = 1$ and $\omega_n = 10$.

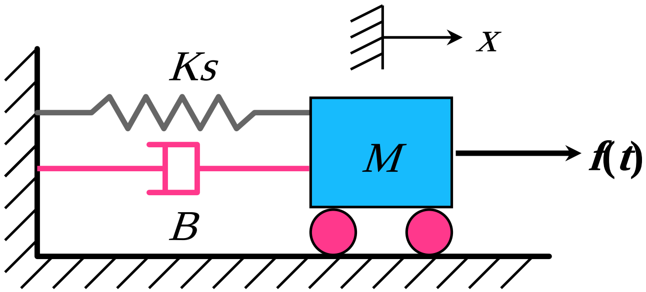
Performance of a Second-Order System



The step response for $\omega_n = 5$ with $\zeta = 0.7$ and $\zeta = 1$.

In Class Exercise

- **Mass-Spring-Damper System**



I/O Model:

Q: *What is the static gain of the system ?*

Q: *How would the physical parameters (M, B, K) affect the step response of the system ?*

(This is equivalent to asking you for the relationship between the physical parameters and the damping ratio, natural frequency and the static gain.)

$$\ddot{y} + a_1\dot{y} + a_0 y = b_0\, u \quad \Rightarrow \quad \ddot{y} + \underbrace{2\zeta\omega_n}_{a_1}\dot{y} + \underbrace{\omega_n^{\,2}}_{a_0} y = \underbrace{K\omega_n^{\,2}}_{b_0} u$$

ω_n : **Natural Frequency [rad/s]**

ζ : **Damping Ratio** $\quad \varsigma = \dfrac{a_1}{2\omega_n} = \dfrac{a_1}{2\sqrt{a_0}}$

K : **Static (Steady State, DC) Gain** $\quad K = \dfrac{b_0}{\omega_n^{\,2}} = \dfrac{b_0}{a_0}$

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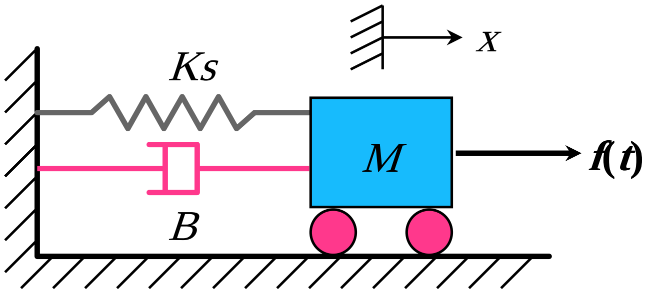
In Class Exercise

- **Mass-Spring-Damper System**



I/O Model:

$$M\ddot{x} + B\dot{x} + K_s x = f(t)$$

$$G(s) = \frac{1}{Ms^2 + Bs + K_s}$$
$$= \frac{1/M}{s^2 + \frac{B}{M}s + \frac{K_s}{M}}$$
$$= \frac{K\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

$$\omega_n = \sqrt{\frac{K_s}{M}}$$
$$\varsigma = \frac{B}{2\sqrt{MK_s}}$$
$$K = \frac{1}{K_s}$$

Q: *What is the static gain of the system ?*

Q: *How would the physical parameters (M, B, K) affect the step response of the system ?*

(This is equivalent to asking you for the relationship between the physical parameters and the damping ratio, natural frequency and the static gain.)

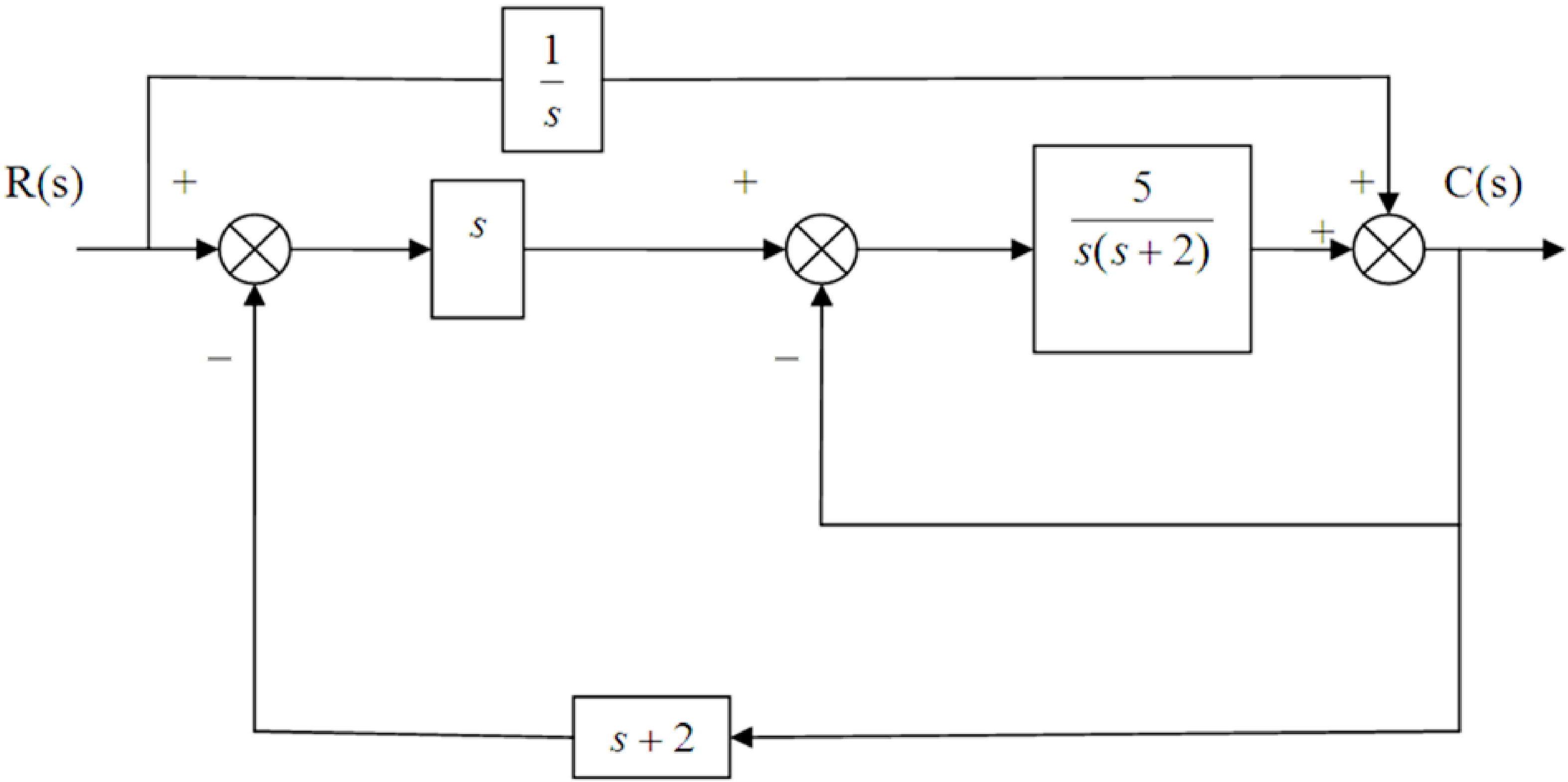
POP. QUIZ

Find for the given control system block diagram:

a) Show that Transfer function is: $\frac{C(s)}{R(s)} = \frac{6s+2}{6s^2+12s+5}$

b) Peak time and OS

c) Static gain, K



POP. QUIZ

$$\frac{C(s)}{R(s)} = \frac{s\dfrac{5}{s(s+2)} + \dfrac{1}{s}}{1 + s\dfrac{5}{s(s+2)}(s+2) + \dfrac{5}{s(s+2)}} = \frac{5s + (s+2)}{s(s+2) + 5s(s+2) + 5} = \frac{6s+2}{6s^2 + 12s + 5}$$

Time domain response specifications

- Defined based on unit step response
- Defined for closed-loop system

- Steady-state value y_{ss}

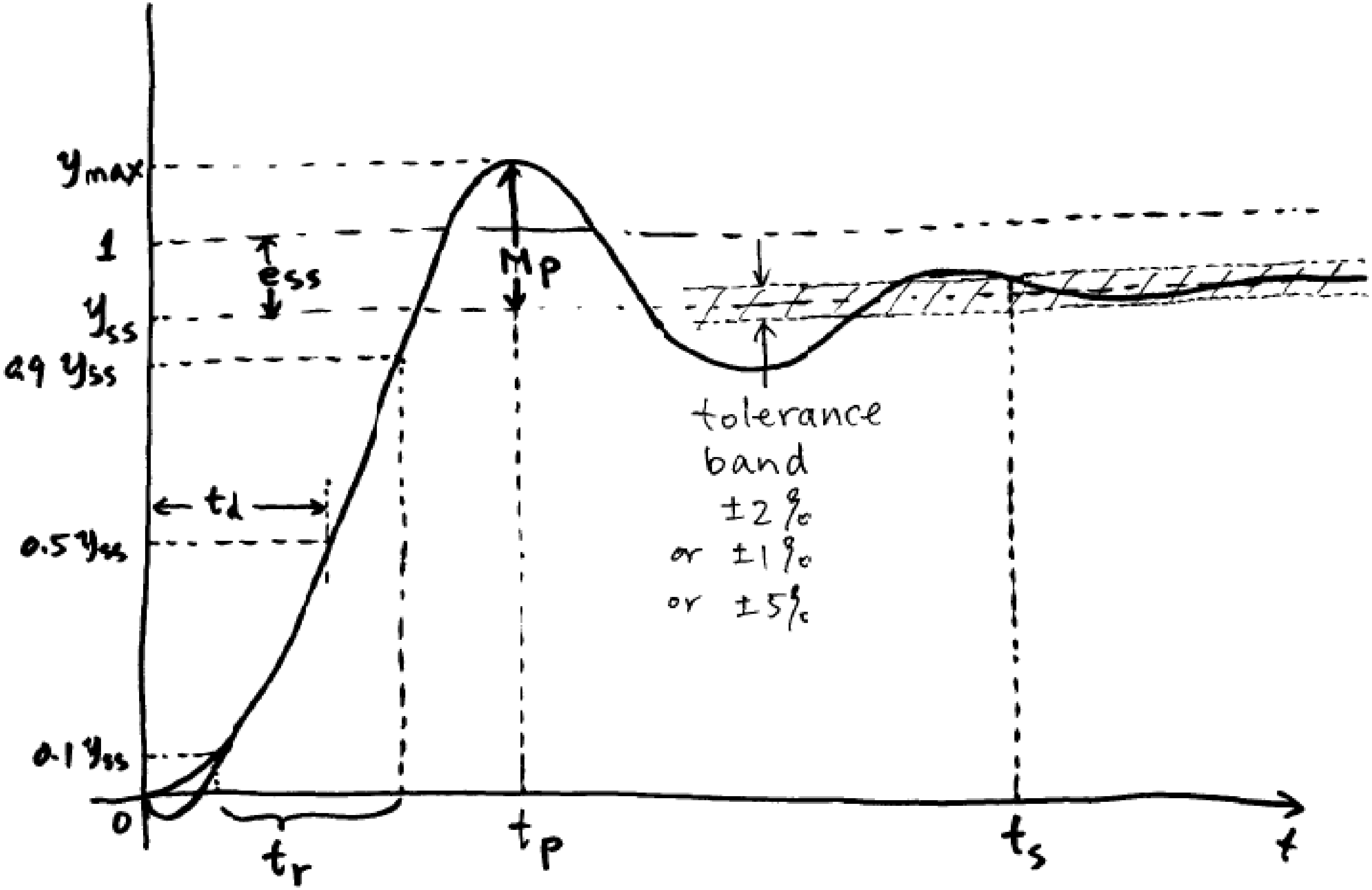
$$= y(\infty) = \lim_{t \to \infty} y(t) \quad , \qquad (\text{input} = u_s(t))$$

- Steady-state error e_{ss}

$$= e(\infty) = \lim_{t \to \infty} e(t) = 1 - y_{ss}$$

- Settling time t_s

  = time when *y(t)* last enters a tolerance band



ymax
1
ess
Mp
yss
0.9 yss
tolerance
band
±2%
or ±1%
or ±5%
td
0.5 yss
0.1 yss
0
tr
tp
ts
t

Peak time t_p = time when $y(t)$ reaches its maximum value

Peak time : $t_p = t(y = y_{\max});$

$$y_{\max} = \max(y);$$

hence : $y_{\max} = y(t_p)$

Overshoot : $M_p = y_{\max} - y_{ss}$

percentage overshoot :

$$M_p = \frac{y_{\max} - y_{ss}}{y_{ss}} 100\% \neq \frac{y_{\max} - 1}{1} 100\%$$

If $y_{\max}$ is reached as $t \to \infty$, there is no peak time & there is no overshoot

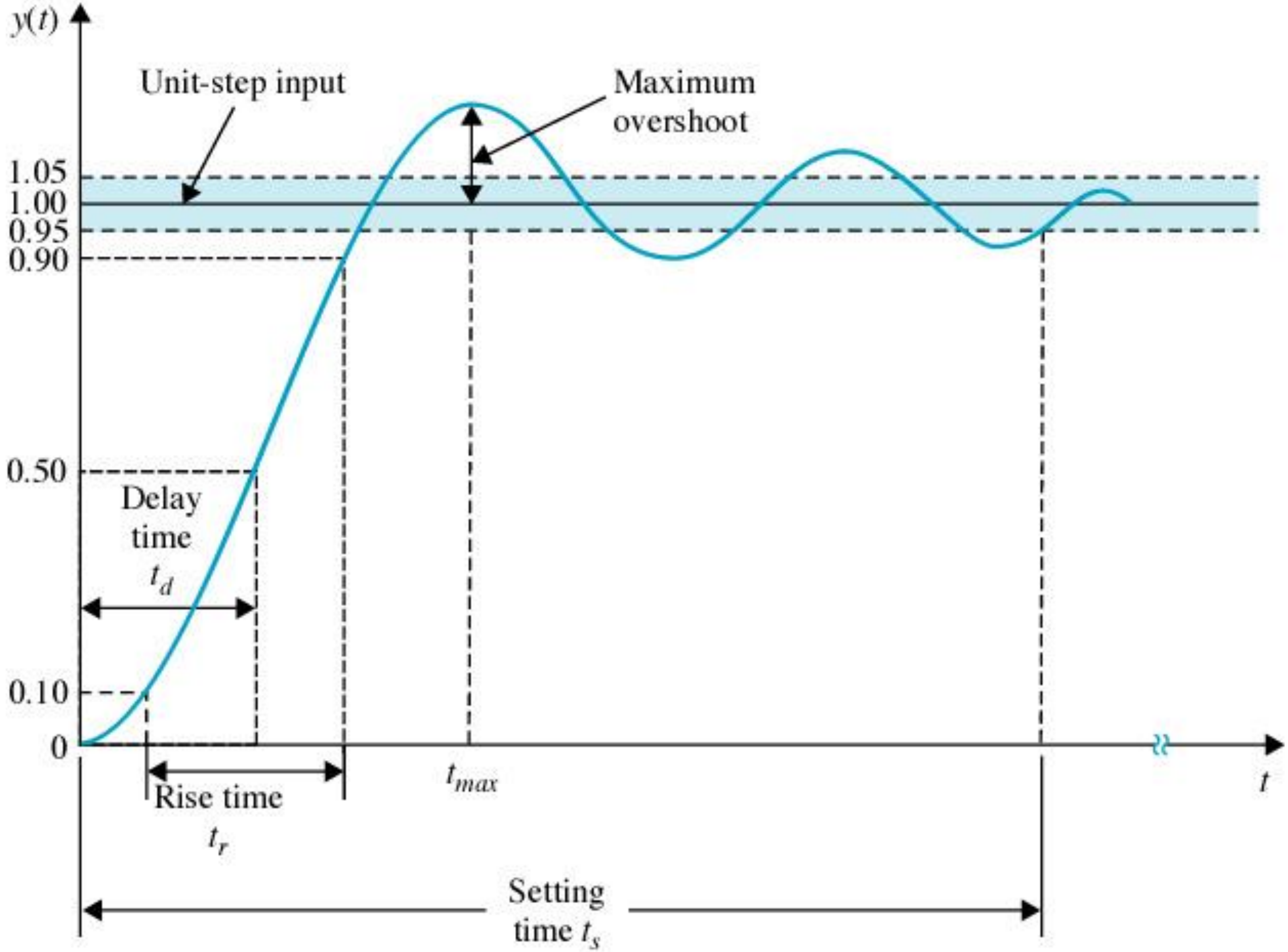
Delay time $t_d =$ the time when $y(t)$ first reaches 50% of y_{ss}

- not freq. used
- some people use a percentage different from 50%

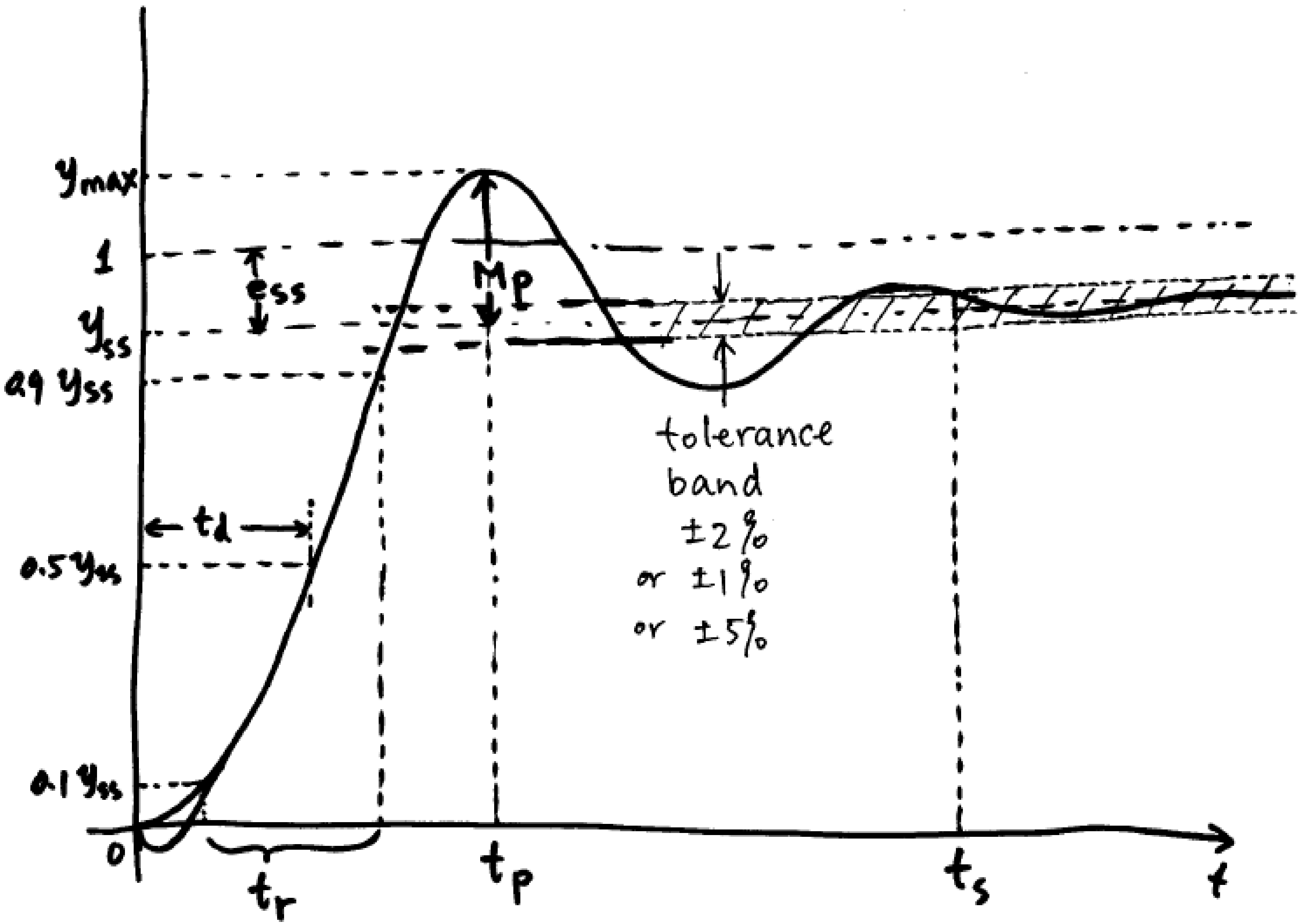
rise time t_r = the time it takes for $y(t)$ to go from $0.1y_{ss}$ to $0.9y_{ss}$ for the first time.

- rise time captures how fast a system responds to changes in reference input

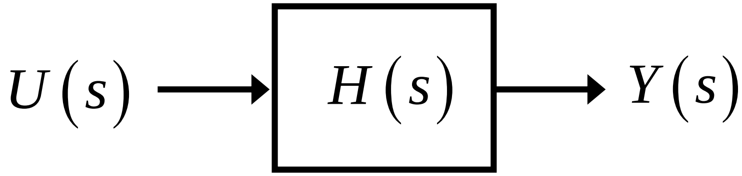
- t_d, t_p has similar effect



y(t)
Unit-step input
Maximum
overshoot
1.05
1.00
0.95
0.90
0.50
0.10
0
Delay
time
t_d
Rise time
t_r
t_{max}
t
Setting
time t_s



ymax
1
ess
Mp
yss
0.9 yss
0.5 yss
0.1 yss
td
tolerance
band
±2%
or ±1%
or ±5%
0
tr
tp
ts
t



$$U(s) = \frac{1}{s}$$

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + \cdots + a_1 s + a_0}$$

$$Y(s) = H(s)\frac{1}{s}$$

By final value theorem

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} H(s) = \frac{b_0}{a_0}$$

In MATLAB: num = [.. .. .. ..]

b0 = num(length(num)), or num(end)

a0 = den(length(den)), or den(end)

yss=b0/a0

$$e_{ss} = 1 - y_{ss}$$

If numerical values of $y(t)$ available,

$abs(y - y_{ss}) < tol$ means inside band

$abs(y - y_{ss}) \geq tol$ not inside

e.g. $t\_out = t(abs(y - y_{ss}) \geq tol)$ contains all those time points when y is not inside the band.

Therefore, the last value in t_out will be the settling time.

```
ts=t_out(end)
```

Peak time t_p = time when $y(t)$ reaches its maximum value.

Peak value $y_{max} = y(t_p)$

Hence: $y_{max} = max(y)$;

$t_p = t(y = y_{max})$;

Overshoot: $OS = y_{max} - y_{ss}$

Percentage overshoot:

$$M_p = \frac{y_{\max} - y_{ss}}{y_{ss}} \cdot 100\% \neq \frac{y_{\max} - 1}{1} \cdot 100\%$$

If $t50 = t(y >= 0.5 \cdot y_{ss})$,

this contains all time points when

$y(t)$ is ≥ 50% of y_{ss}

so the first such point is t_d.

```
td=t50(1);
```

Similarly, $t10 = t(y <= 0.1 * y_{ss})$

& $t90 = t(y >= 0.9 * y_{ss})$

can be used to find t_r.

```
tr=t90(1)-t10(end)
```

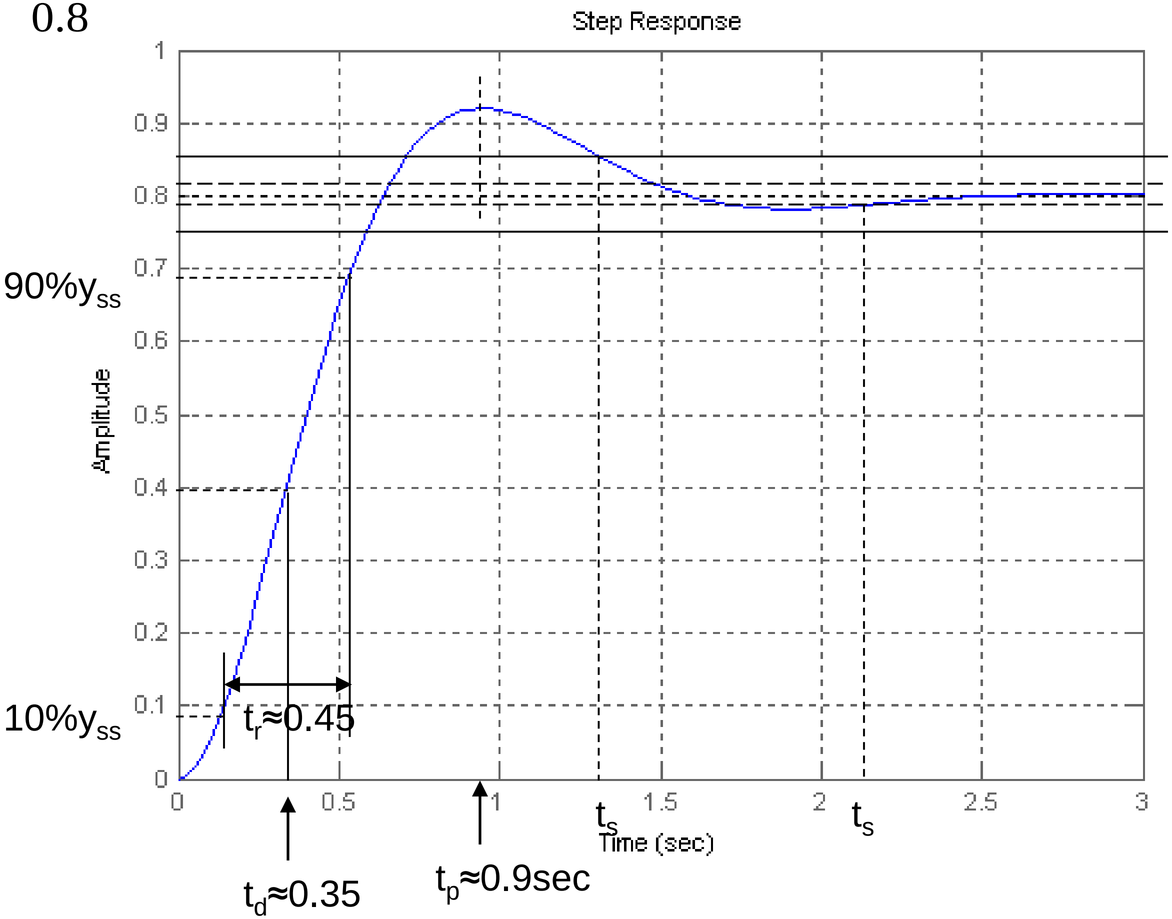
specs are defined on step resp. $\begin{cases} u = \text{step} \\ \text{i.c.} = 0 \end{cases}$

$$y_{ss} = y(\infty) = 0.8, y_{d.s.s.} = 1$$

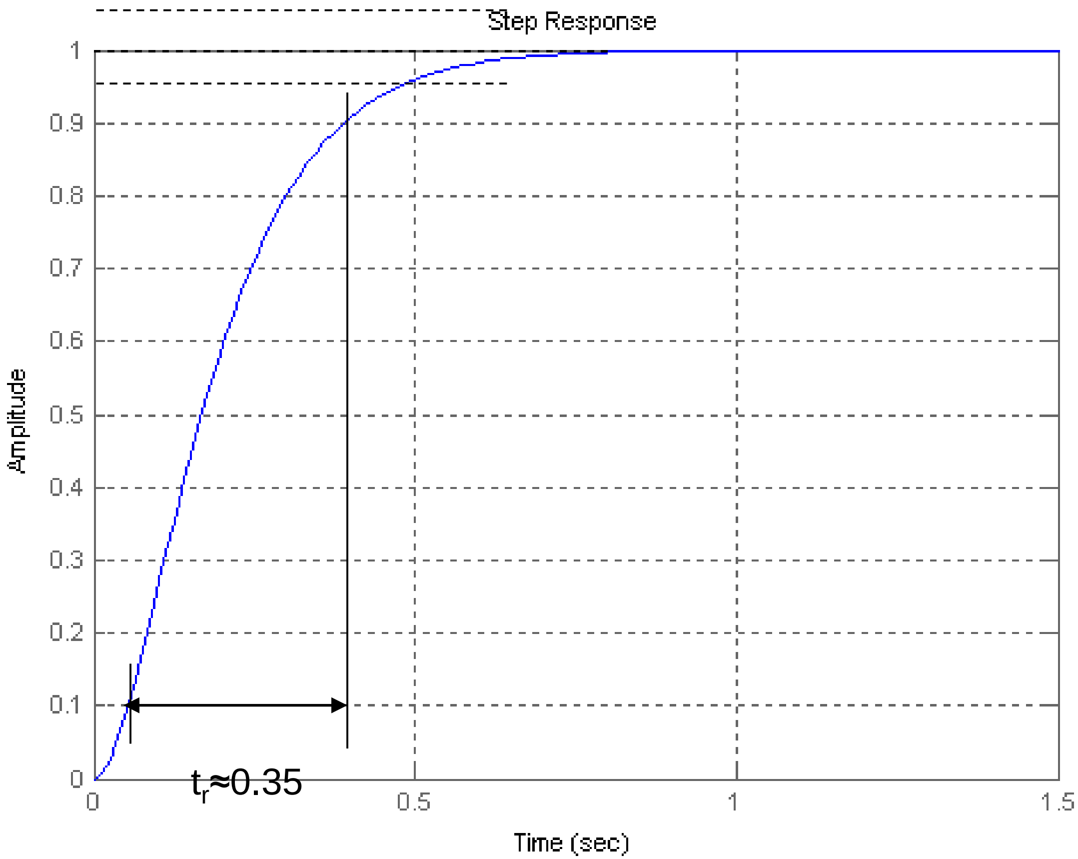
$$e_{ss} = 0.2 = y_{d.} - y_{ss}$$

$$y_{\max} = 0.92, \text{ overshoot } = 0.92 - 0.8 = 0.12$$

$$\text{percentage o.s.} = \frac{0.12}{0.8} = 15\%$$



±5% $t_s=0.45$



$y_{ss}=1$

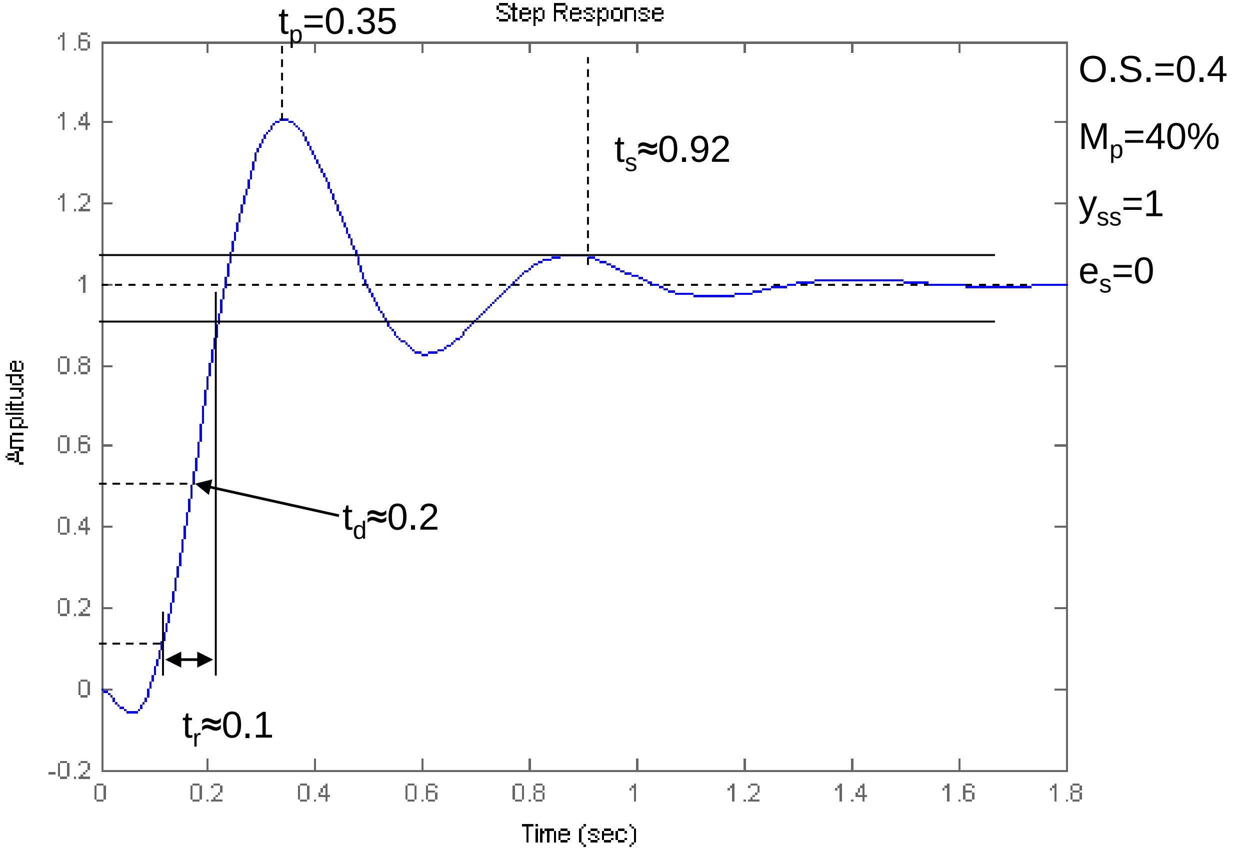
$e_{ss}=0$

O.S.=0

$M_p=0$

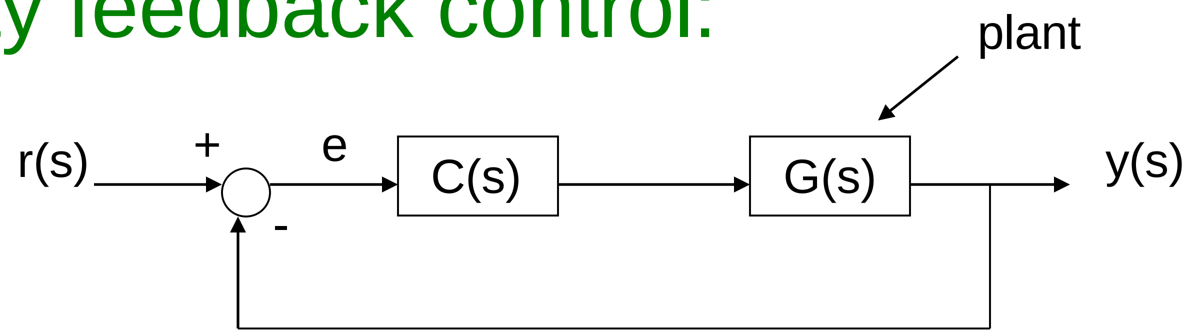
$t_p=\infty$

$t_d \approx 0.2$

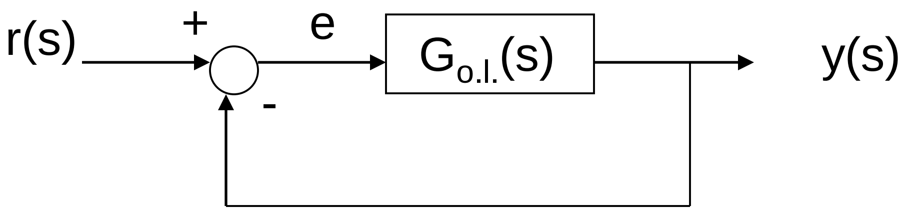

Step Response
t_p=0.35
O.S.=0.4
M_p=40%
t_s≈0.92
y_ss=1
e_s=0
t_d≈0.2
t_r≈0.1
Amplitude
Time (sec)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8

Steady-state tracking & sys. types

- Unity feedback control:



$\Longrightarrow$



$$\frac{y(s)}{r(s)} = \frac{G_{ol}}{1+G_{ol}}$$

$\dfrac{y(s)}{e(s)} = G_{o.l.}(s)$ is the open loop T.F. from e to y

i.e. cut loop open, & get T.F.

$\rightarrow G_{o.l.}$ can always be factored into :

$$G_{o.l.} = \frac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

$$= \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_{N+1}s^{N+1} + a_N s^N + \cdots + a_1 s + a_0}$$

$N \geq 0$ $\qquad$ $a_{N+1} \cdots a_1, a_0 = 0$

$N + p = n$ $\qquad$ but $a_N \neq 0$

$K = \frac{b_0}{a_N}$ $\qquad$ $b_m \neq 0$

If $N \neq 0$, need $b_0 \neq 0$, otherwise can cancel an s.

- closed-loop: $y(s) = \dfrac{G_{o.l.}(s)}{1+G_{o.l.}(s)} r(s)$

- tracking error: $e(s) = r(s) - y(s) = \dfrac{1}{1+G_{o.l.}(s)} r(s)$

- steady-state tracking:

$$e(t=\infty) = e_{ss} = \lim_{s\to 0} se(s) = \left.\frac{sr(s)}{1+G_{o.l.}(s)}\right|_{s=0}$$

- For step input: $r(s) = \dfrac{1}{s}$

$$e_{ss} \text{ to step} = \left.\frac{1}{1+G_{o.l.}(s)}\right|_{s=0} = \frac{1}{1+K_p}$$

denote $\boxed{K_p = \lim_{s \to 0} G_{o.l.}(s) = G_{o.l.}(0)}$

↑

called static position error const.

Then $\boxed{e_{ss} \text{ to step} = \dfrac{1}{1+K_p}}$

(here use small p, not to be confused with proportional control K_P)

- If $N = 0$, the system is called "type 0" with respect to r,

$$K_p = G_{o.l.}(0) = \frac{b_0}{a_0} \quad \text{finite}$$