MENG366

Routh's Stability Criterion

Part I

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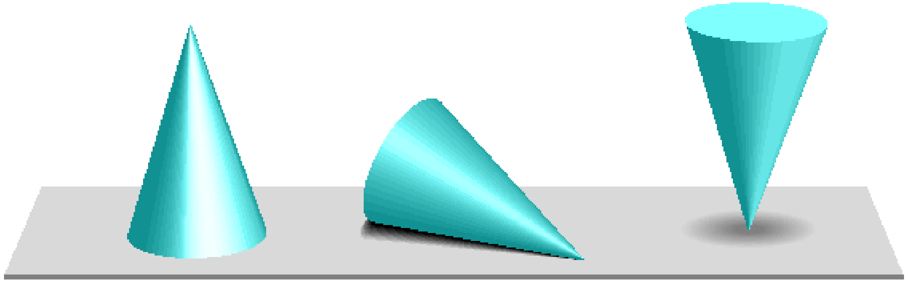
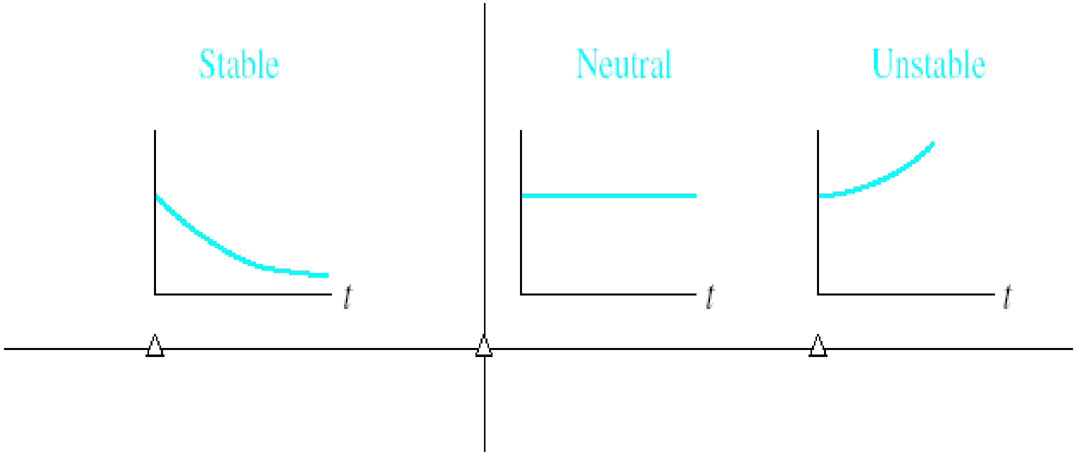
The Concept of Stability

The concept of stability can be illustrated by a cone placed on a plane horizontal surface.



(a) Stable (b) Neutral (c) Unstable



A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

A system is considered marginally stable if only certain bounded inputs will result in a bounded output.

Stability

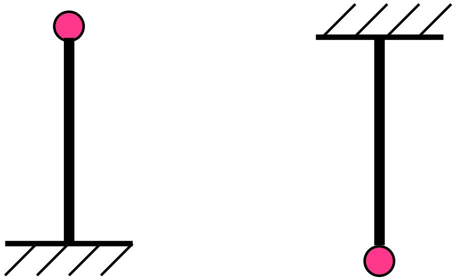
where the derivatives of all states are zeros

- **Stability Concept**

  Describes the ability of a system to stay at its equilibrium position in the absence of any inputs.

  - **A linear time invariant (LTI) system is stable if and only if (iff) its free response converges to zero for all ICs.**

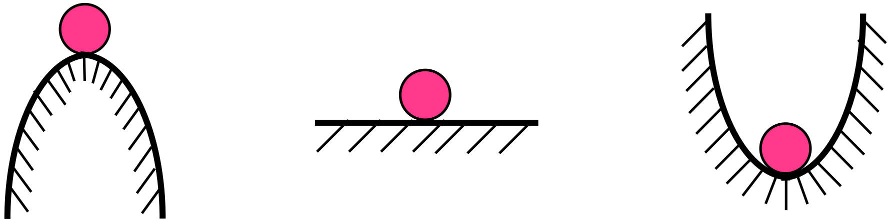
Ex: Pendulum



inverted pendulum

simple pendulum

Ball on curved surface



hill

plateau

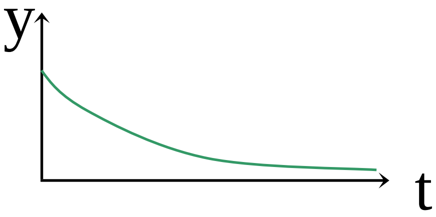
valley

Examples (stable and unstable 1st order systems)

Q: free response of a 1st order system.

$$5\dot{y} + y = u(t) \qquad y(0) = y_0$$

$$\boxed{y(t) = y_0 e^{-\frac{1}{5}t}}$$



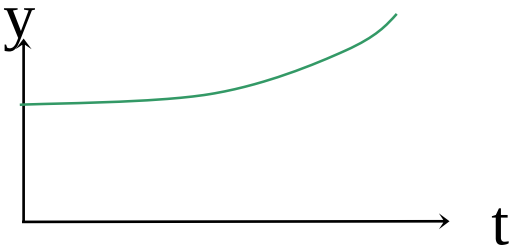
TF: $G = \dfrac{1}{5s+1}$

Pole: $p = -0.2$

Q: free response of a 1st order system.

$$-5\dot{y} + y = u(t) \qquad y(0) = y_0$$

$$\boxed{y(t) = y_0 e^{\frac{1}{5}t}}$$



TF: $G = \dfrac{1}{-5s+1}$

Pole: $p = 0.2$

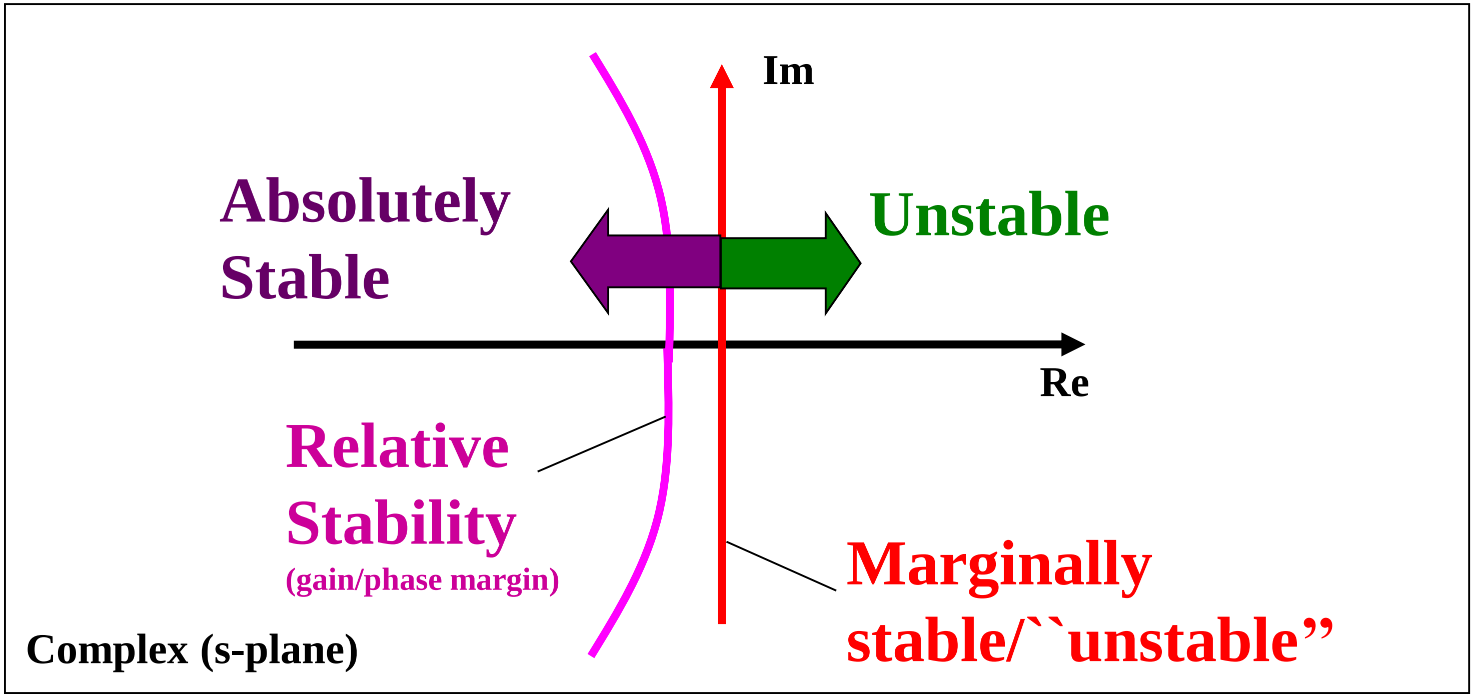
$$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$$

Stability of LTI Systems

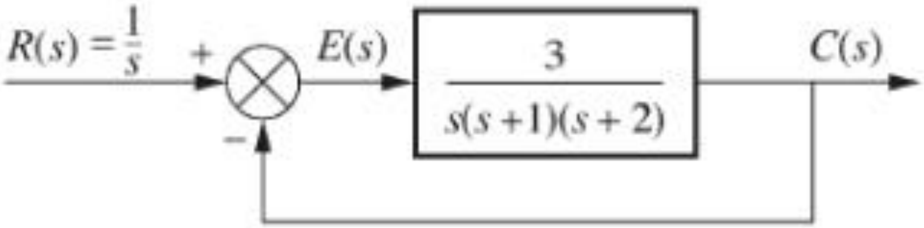
$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_1\dot{u} + b_0 u$$

Stable $\Longleftrightarrow$ All poles lie in the left-half complex plane (LHP)

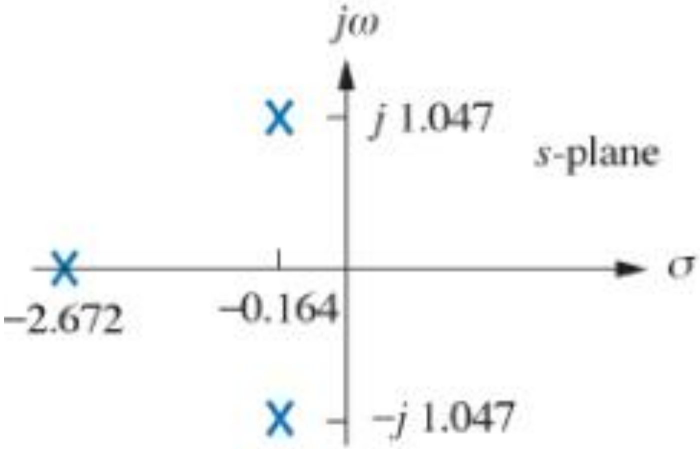
$\Longleftrightarrow$ All roots of $D(s) = \underbrace{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}_{\text{Characteristic Polynomial}} = 0$ lie in the LHP



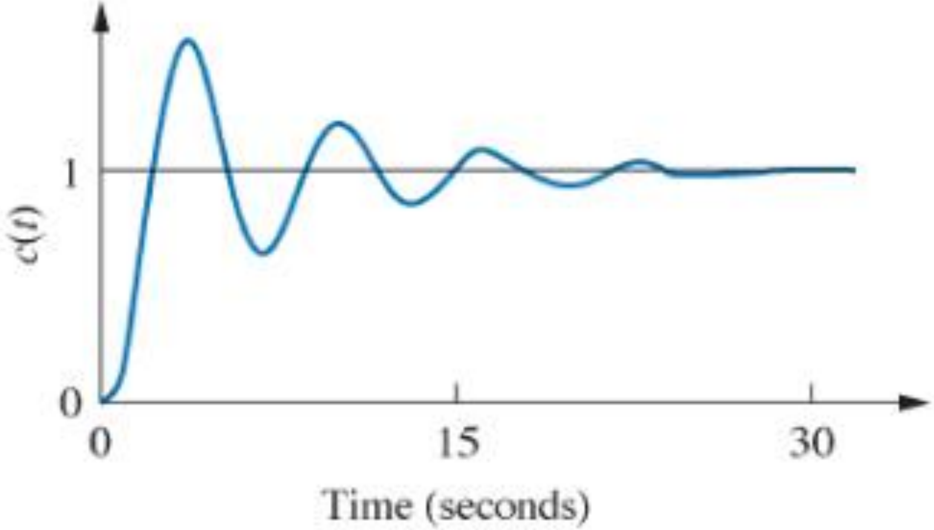
Stable System



Stable system



Stable system's closed-loop poles (not to scale)



Stable System

Closed –Loop TF

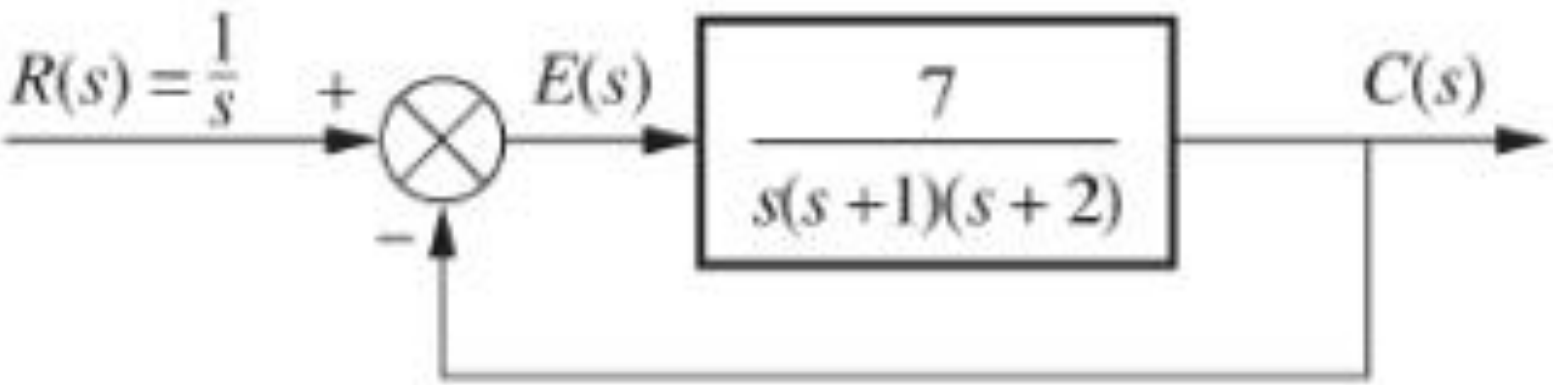
$$G = \frac{3}{s^3 + 3s^2 + 2s + 3}$$

Poles

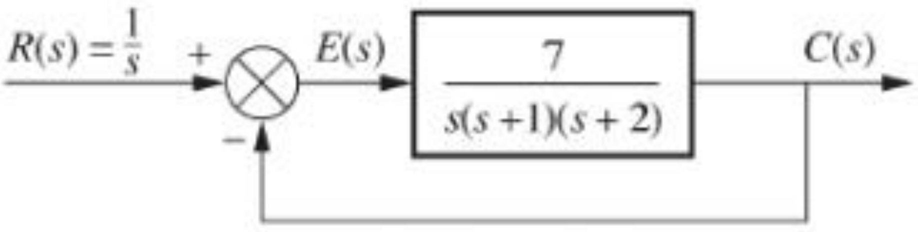
```
>> p=[1 3 2 3];
>> roots(p)

ans =


  -2.6717
  -0.1642 + 1.0469i
  -0.1642 - 1.0469i
```

All in LHS of s- plane

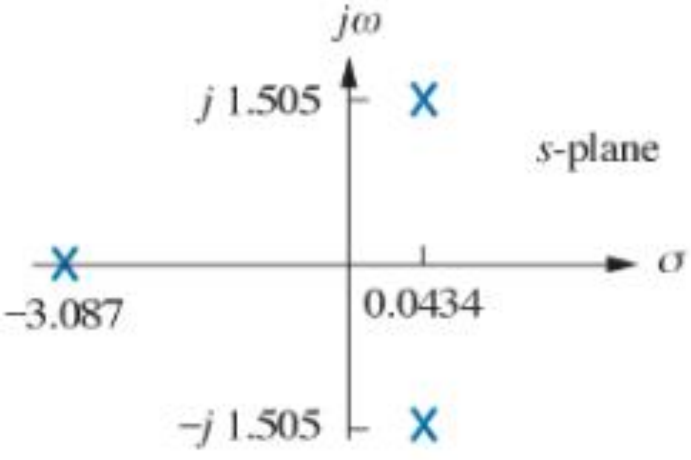
Teamwork Example



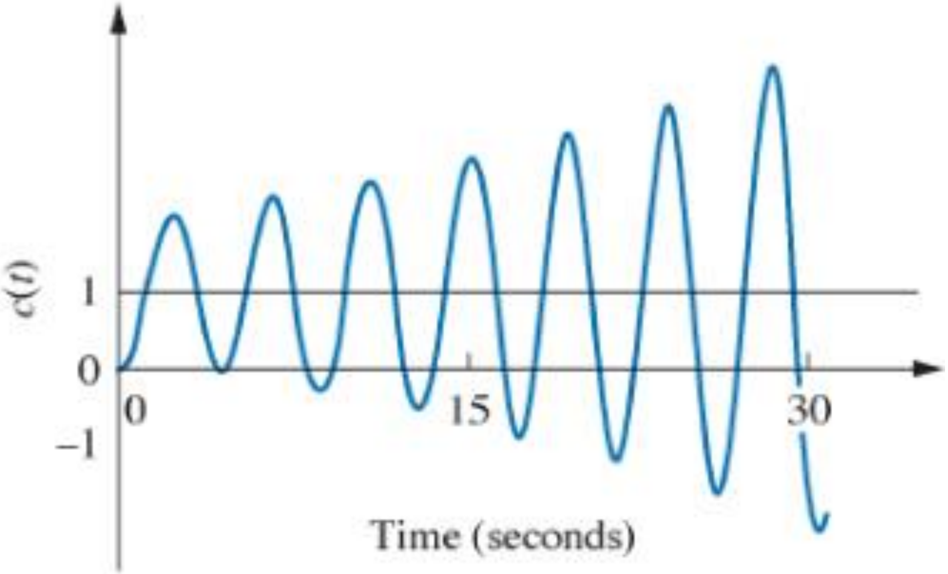
Unstable System



Unstable system



Unstable system's closed-loop poles (not to scale)



Unstable System

Closed –Loop TF

$$G = \frac{7}{s^3 + 3s^2 + 2s + 7}$$

Poles

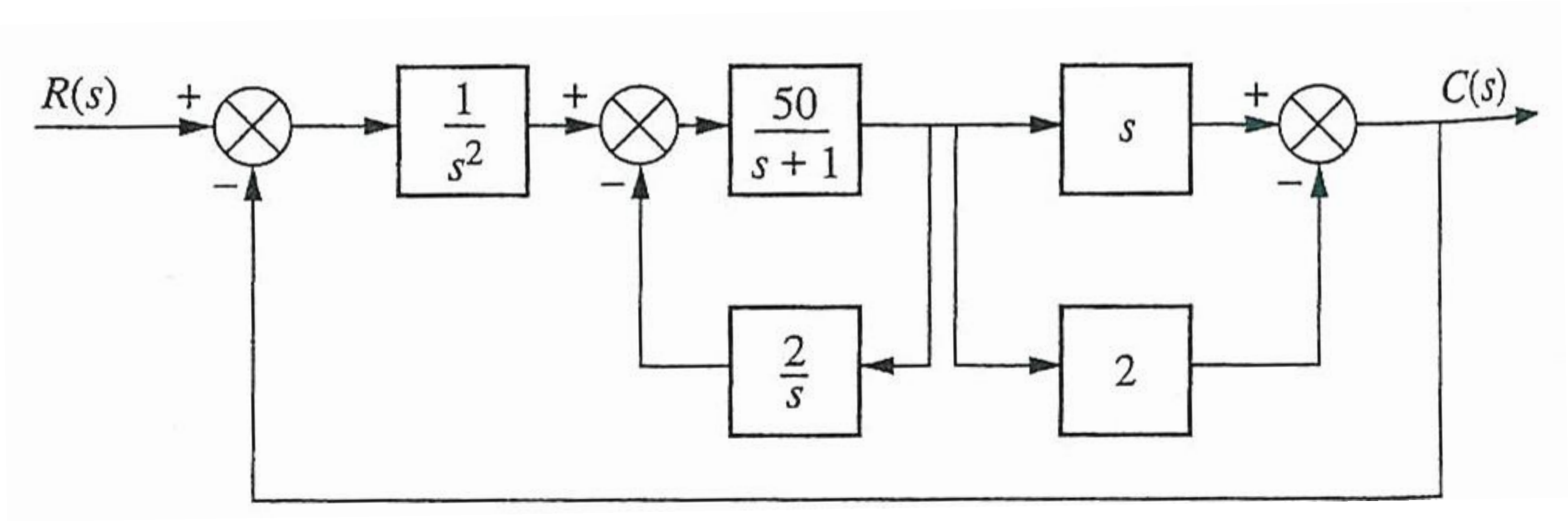
```
>> p=[1 3 2 7];
>> roots(p)

ans =


   -3.0867
   0.0434 + 1.5053i
   0.0434 - 1.5053i
```

One in LHS of s- plane

Two in RHS

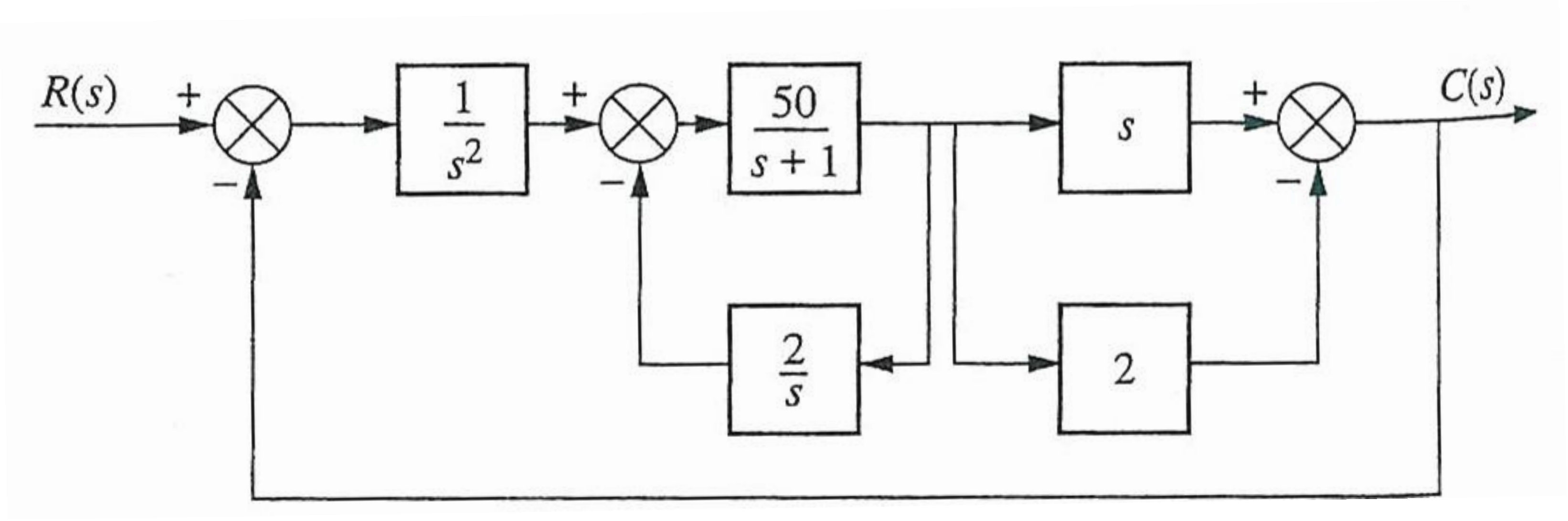
POP. Quiz

Block Diagram



Unstable System

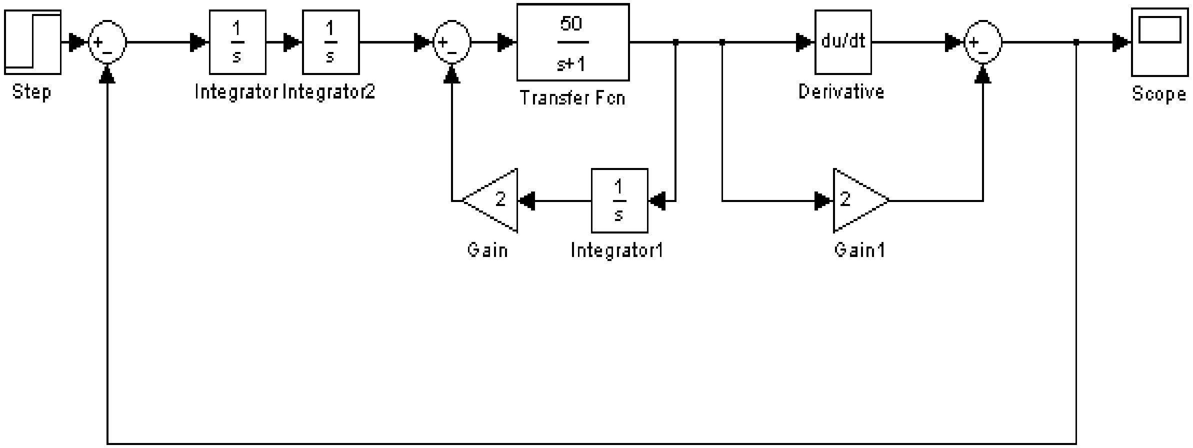
Block Diagram

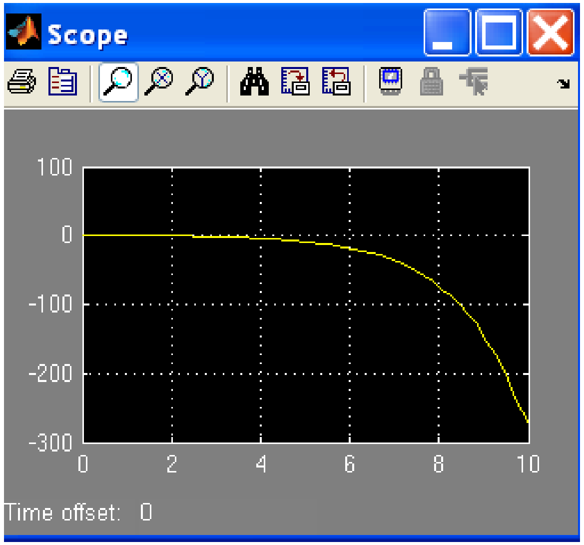


$$T(s) = \frac{50(s-2)}{s^3+s^2+150s-100}$$

Unstable System

Simulink model

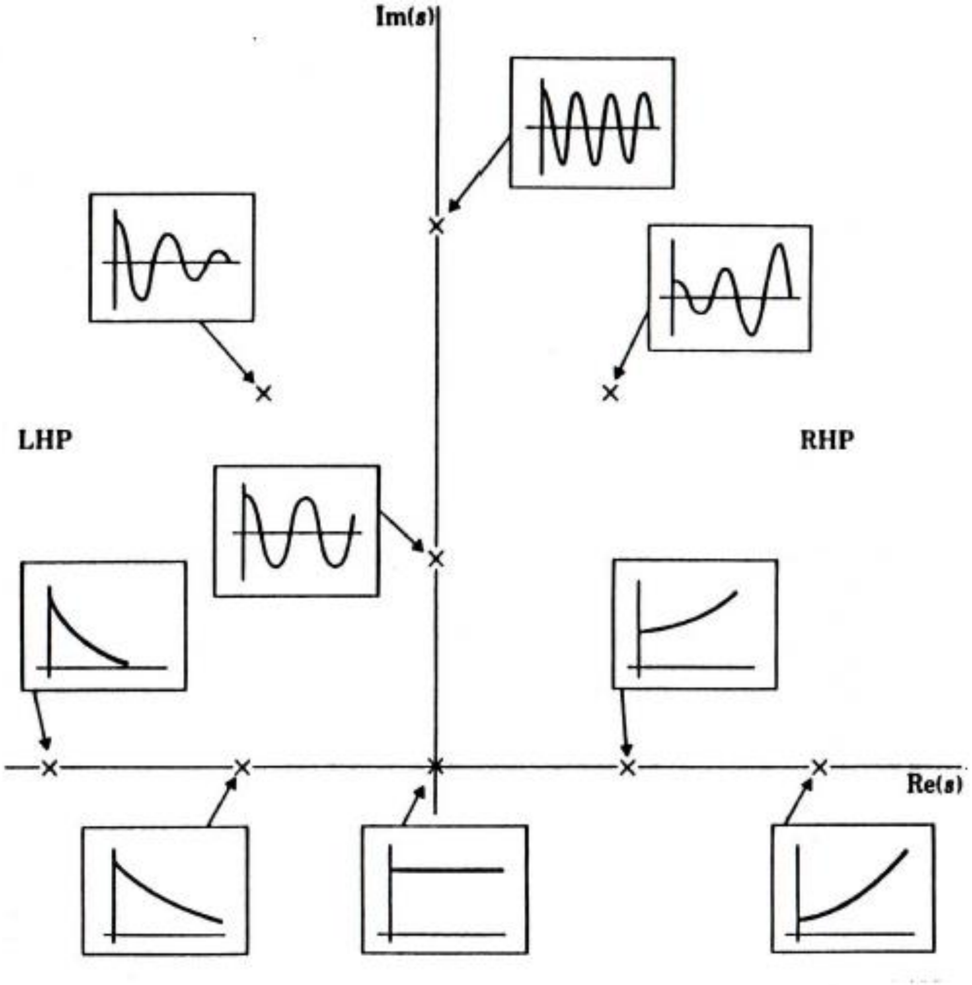




Unstable as $T(s) = \dfrac{50(s-2)}{s^3+s^2+150s-100}$

has poles at -0.8309 ±12.2642i, 0.6618

Stable & Unstable System



Stability Definitions

- **Bounded Input Bounded Output Stability:**

  BIBO system is stable if, for every bounded input, the output remains bounded with increasing time (**all system poles must lie in the left half of the s-plane**).

- **Marginal Stability:**

  Occurs if some of the poles lie on the **imaginary axis**, while all others are in the LHS of the s-plane.

Internal (Asymptotic) Stable

Definition :

A system is internal (asymptotic) stable, if the zero-input response decays to zero, as time approaches infinity, for all possible initial conditions.

*Asymptotic stable =>*All the characteristic polynomial roots are located in the LHP (left-half-plan)

External (BIBO) Stable

Definition :

A system is external (bounded-input, bounded-output, BIBO) stable, if the zero-state response is bounded, as time approaches infinity, for all bounded inputs.

bounded-input, bounded-output stable =>All the poles of transfer function are located in the LHP (left-half-plan)

Asymptotic stable => BIBO stable
BIBO stable≠> Asymptotic stable

Poles and zeros

- Poles: values of s at which TF → infinity
  - Most time, poles = roots of denominator
  - When there are common factors in numerator and denominator, cancel them first
- Zeros: values of s at which TF = 0
  - Finite zeros: roots of numerator
  - Number of zeros at infinity: n-m, n= den deg and m = num deg
- Totally n poles and n zeros
- n is called the order of the system
- n – m the relative order

Example

$$G(s) = \frac{10(s+1)}{s^2(s+4)(s+6)}$$

- Order: n = 4 = den deg
- m= num deg = 1
- Relative order = n-m = 3
- 4 poles at: 0, 0, -4, -6
- One finite zero at -1
- 3 zeros at infinity

BIBO Stability

- System is BIBO stable if any bounded input generates bounded output
- Simple criteria:
  - After common factor cancellation
  - All poles have strictly negative real parts

BIBO Stability

POP. Quiz:

$$G_1(s) = \frac{10(s+1)}{s^2(s+4)(s+6)}$$

$$G_2(s) = \frac{10s+3}{(s+2)(s+5)}$$

$$G_3(s) = \frac{s-1}{s^2+4s+6}$$

$$G_4(s) = \frac{e^{-2s}}{10s(s+1)(s+2)}$$

BIBO Stability

POP. Quiz:

$$G_1(s)=\frac{10(s+1)}{s^2(s+4)(s+6)}$$

$$G_2(s)=\frac{10s+3}{(s+2)(s+5)}$$

$$G_3(s)=\frac{s-1}{s^2+4s+6}$$

$$G_4(s)=\frac{e^{-2s}}{10s(s+1)(s+2)}$$

G1 and G4 are not BIBO stable

G2 and G3 are BIBO stable

Stability Analysis

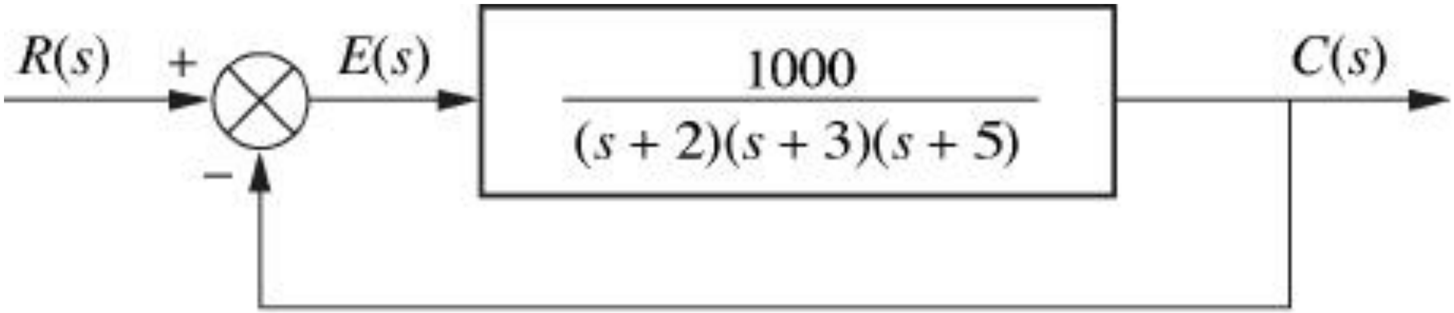
Methods for Testing Stability of a LTI system

1. Examine the poles of the system.

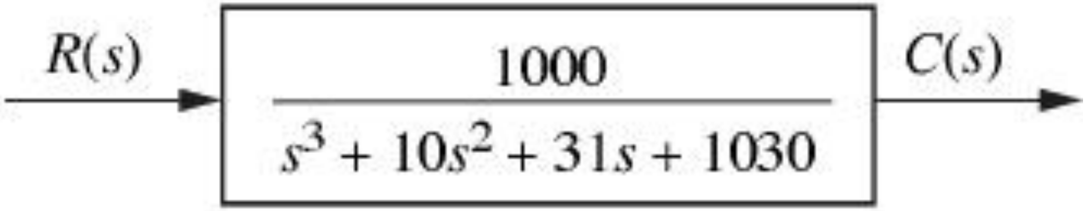
2. Methods the do not require the actual solution of the characteristic equation.

3. Methods that are based only on the loop transfer function characteristics.

Poles of Complex Systems



Open-loop poles: s= -2, -3, -5



Closed-loop poles: -13.4136, 1.7068 + 8.5950*j*, 1.7068 - 8.5950*j*

Routh-Hurwitz Criterion

Is a method that checks system stability without actual solution of the characteristic equation.

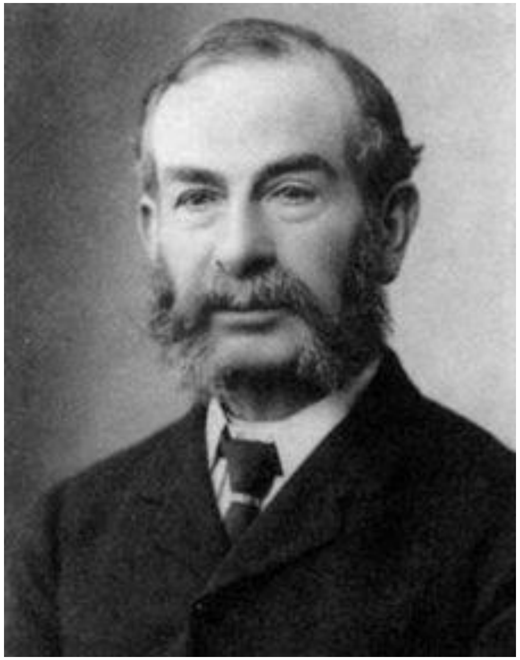
Is a method for establishing bounds on system parameters to ensure stability.

Nuoki Mutsumoto, "Simple Proof of The Routh Stability Criterion Based on Order Reduction of Polynomials and Principle of Argument", *The 2001 IEEE International Symposium on Circuits and Systems*, 6-9 May, 2001, Sydney Australia, Vol. I, pp. 699-702

E. Routh & A. Hurwitz

Edward John Routh

(1831-1907)



University of Cambridge

Adolf Hurwitz

(1859 -1919)

Eidgenössische Polytechnikum Zürich

Routh-Hurwitz Stability Criterion

A quick method for checking BIBO stability

- Assume the characteristic polynomial is

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$$

where $a_0 \neq 0$

- A **necessary** (but not sufficient) condition for all roots to have non-positive real parts is that **all coefficients have the same sign.**
- For the **necessary and sufficient** conditions, the sign of the first column of the **Routh array** should not change.

The Routh Array

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$$

$$\begin{array}{ccccc}
a_n & a_{n-2} & a_{n-4} & a_{n-6} & \ldots \\
a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \ldots \\
b_1 & b_2 & b_3 & b_4 & \ldots \\
c_1 & c_2 & c_3 & c_4 & \ldots \\
\vdots & \vdots & \vdots & & \\
k_1 & k_2 & & & \\
l_1 & & & & \\
m_1 & & & &
\end{array}$$

where

$$b_1 = \frac{a_{n-1} a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1} a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

etc.

The Routh Array

- In a similar manner, elements in the 4th row, c_1, c_2 , … are calculated based as:

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$$

$$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

- The elements in all subsequent rows are calculated in the same manner.

Routh-Hurwitz Stability Criterion

Necessary and sufficient conditions:

- If all elements in the **first column** of the Routh array have the **same sign**, then all roots of the characteristic equation have negative real parts.
- If there are sign changes in these elements, then the number of roots with non-negative real parts is equal to the **number of sign changes**.
- Elements in the first column which are **zero** define a special case.

Proof of Routh-Hurwitz Criterion

Second Order Systems

$$Q(s) = s^2 + as + b = 0$$

with *a, b* real & positive

Routh Table

s^2	1	b
s^1	a	0
s^0	b_1	

$$b_1 = \frac{a \times b - 1 \times 0}{a} = b$$

Necessary Condition

System is stable if ***a & b*** are positive, i.e. no sign changes

Proof of Routh-Hurwitz Criterion

Note that the roots of the CE are:

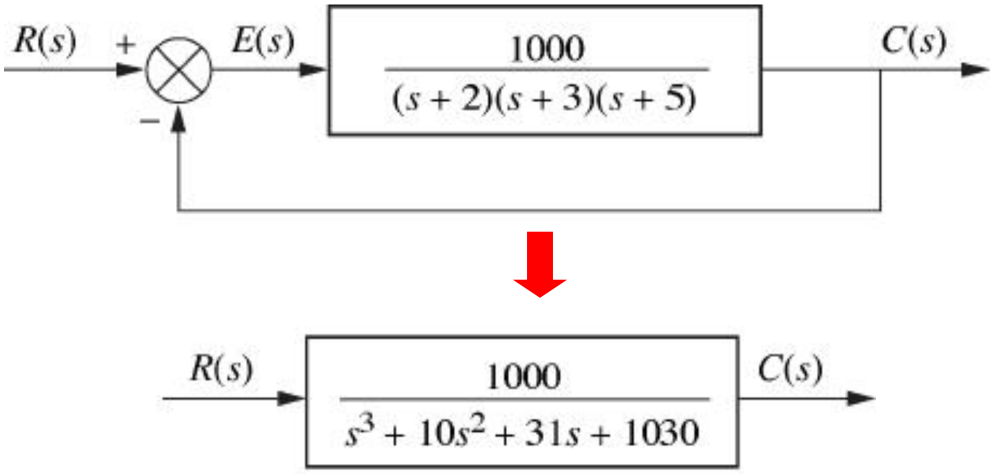
$$s_{1,2} = \frac{-a \pm \sqrt{-a^2 + 4b}\,j}{2}$$

$$\Rightarrow \qquad s_1 + s_2 = -a, \; s_1 s_2 = b$$

Sufficient Condition

Or *a & b* are positive only when the roots have negative real parts

Example 6.1



s^3	1	31	0
s^2	$\cancel{10}\ \ 1$	$\cancel{1030}\ \ 103$	0
s^1	$\frac{-\begin{vmatrix}1 & 31\\ 1 & 103\end{vmatrix}}{1} = -72$	$\frac{-\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}}{1} = 0$
s^0	$\frac{-\begin{vmatrix}1 & 103\\ -72 & 0\end{vmatrix}}{-72} = 103$	$\frac{-\begin{vmatrix}1 & 0\\ -72 & 0\end{vmatrix}}{-72} = 0$	$\frac{-\begin{vmatrix}1 & 0\\ -72 & 0\end{vmatrix}}{-72} = 0$

Example 6.1

s^3	1	31	0	0
s^2	10	1030	0	
s^1	-72	0		
s^0	1030			

Two sign changes → two roots with positive real parts

```
>> p=[1 10  31
1030];
>> roots(p)

ans =
 -13.4136
   1.7068 + 8.5950i
   1.7068  - 8.5950i
```

Example 6.1

```
>> Routh_H([1  10  31 1030])
--------------------------------------------------------------------------------
                              RESULTS
--------------------------------------------------------------------------------
   Not satisfied with the sufficient condition.
   The system is unstable and has 2 pole(s) in the RSP.
--------------------------------------------------------------------------------


  ROUTH-HURWITZ ARRAY

   s^3  |      1        |      31     |
   s^2  |      10       |     1030   |
   s^1  |     -72       |
   s^0  |     1030    |
```

http://www.mathworks.fr/matlabcentral/fileexchange/19872

Routh-Hurwitz Stability Criterion

Example 1 $Q(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$

Routh-Hurwitz Stability Criterion

Example 1 $$Q(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$$

s^4	2	3	10	0
s^3	1	5	0	0
s^2	b_1	b_2	0	
s^1	c_1	0		
s^0	d_1			

$$b_1 = \frac{3-10}{1} = -7 \qquad b_2 = \frac{10-0}{1} = 10$$

$$c_1 = \frac{-35-10}{-7} = 6.43$$

$$d_1 = \frac{10(6.43)-0}{6.43} = 10$$

The characteristic equation has **two** roots with **positive real parts** since the elements of the first column have two sign changes. (**2,1,-7,6.43,10**)

Routh-Hurwitz Stability Criterion

Roots

$$Q(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$$

```
>> p=[2 3 5 10];
>> roots(p)

ans =

  -1.7279
   0.1139 + 1.6973i
   0.1139 - 1.6973i
```

The characteristic equation has **two** roots with **positive real parts** since the elements of the first column have two sign changes. (**2,1,-7,6.43,10**)

Routh-Hurwitz Stability Criterion

```
>> Routh_H([2  1  3  5 10])
---------------------------------------------------------------------
                        RESULTS
---------------------------------------------------------------------
  Not satisfied with the sufficient condition.
  The system is unstable and has 2 pole(s) in the RSP.
---------------------------------------------------------------------

 ROUTH-HURWITZ ARRAY

  s^4  |    2       |    3    |    10    |
  s^3  |    1       |    5    |
  s^2  |    -7      |    10   |
  s^1  |  6.4286    |
  s^0  |    10      |
```

Routh-Hurwitz Stability Criterion

Special Case 1

- A **zero** in the first column:
- Remedy: substitute **ε** for the zero element, finish the Routh array, and then let **ε → 0** .

$$Q(s) = s^3 - 3s + 2$$

$$b_1 = \frac{-3\varepsilon - 2}{\varepsilon} \rightarrow \frac{-2}{\varepsilon} \text{(negative)}$$

s^3	1	-3	0	0
s^2	$0(\varepsilon)$	2	0	
s^1	b_1	0		
s^0	c_1			

$$c_1 = \frac{b_1 \times 2}{b_1} = 2$$

There are **two** roots with positive real parts (**1, ε, -2/ε , 2**)

Routh-Hurwitz Stability Criterion

Special Case 1

$$Q(s) = s^3 - 3s + 2$$

Routh-Hurwitz Stability Criterion

Special Case 1

$$Q(s) = s^3 - 3s + 2$$

```
>> p=[1 0 -3 2];
>> roots(p)

ans =

  -2.0000
   1.0000
   1.0000
```

There are **two** roots with positive real parts (**1, ε, -2/ε , 2**)

Routh-Hurwitz Stability Criterion

```
>> Routh_H([1 0 -3 2])
-------------------------------------------------------------------------
                          RESULTS
-------------------------------------------------------------------------
   Not satisfied with the necessary condition.
   Not satisfied with the sufficient condition.
   The system is unstable and has 2 pole(s) in the RSP.
-------------------------------------------------------------------------

 ROUTH-HURWITZ ARRAY

   s^3 |     1      |     -3     |
   s^2 |     0      |     2      |
   s^1 |    -Inf    |
   s^0 |     2      |
```

Example on Routh Stability

Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

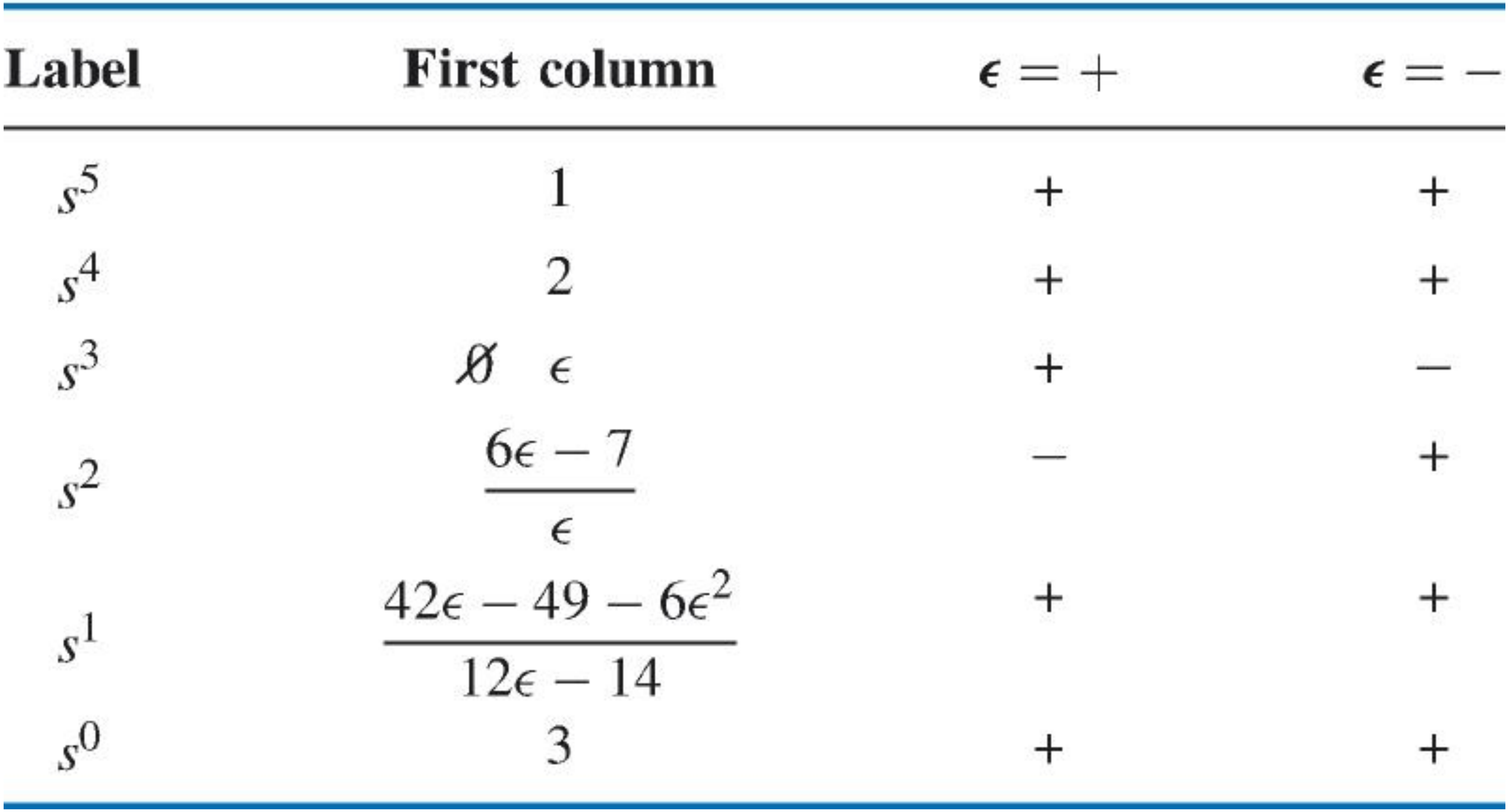
Example on Routh Stability

Routh Table

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0}\quad\epsilon$	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon-7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$	0	0
s^0	3	0	0

Example on Routh Stability

Routh Table



Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	$+$	$+$
s^4	2	$+$	$+$
s^3	$\cancel{0} \quad \epsilon$	$+$	$-$
s^2	$\dfrac{6\epsilon - 7}{\epsilon}$	$-$	$+$
s^1	$\dfrac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	$+$	$+$
s^0	3	$+$	$+$

Example on Routh Stability

Interpretation of Results

Two sign changes → two roots with positive real parts

```
>> p=[1 2 3  6 5  3];
>> roots(p)

ans =


   0.3429 + 1.5083i
   0.3429 - 1.5083i
  -1.6681
  -0.5088 + 0.7020i
  -0.5088 - 0.7020i
```

Example on Routh Stability

```
>> Routh_H([1 2  3 6  5  3])
---------------------------------------------------------------------
                        RESULTS
---------------------------------------------------------------------
  Not satisfied with the sufficient condition.
  The system is unstable and has 2 pole(s) in the RSP.
---------------------------------------------------------------------


 ROUTH-HURWITZ ARRAY

  s^5 |     1     |     3     |     5     |
  s^4 |     2     |     6     |     3     |
  s^3 |     0     |    3.5    |
  s^2 |    -Inf    |     3     |
  s^1 |    3.5    |
  s^0 |     3     |
```

Routh-Hurwitz Stability Criterion

```
>> Routh_H([1 -6  -7  -52])
---------------------------------------------------------------------
                        RESULTS
---------------------------------------------------------------------
   Not satisfied with the necessary condition.
   Not satisfied with the sufficient condition.
   The system is unstable and has 1 pole(s) in the RSP.
---------------------------------------------------------------------


  ROUTH-HURWITZ ARRAY

   s^3 |     1         |    -7    |
   s^2 |     -6        |    -52   |
   s^1 |  -15.6667     |
   s^0 |    -52        |
```

System Response of n^{th} Order

(i) First order system response

(ii) Second order system response

(iii) High order system response

First order

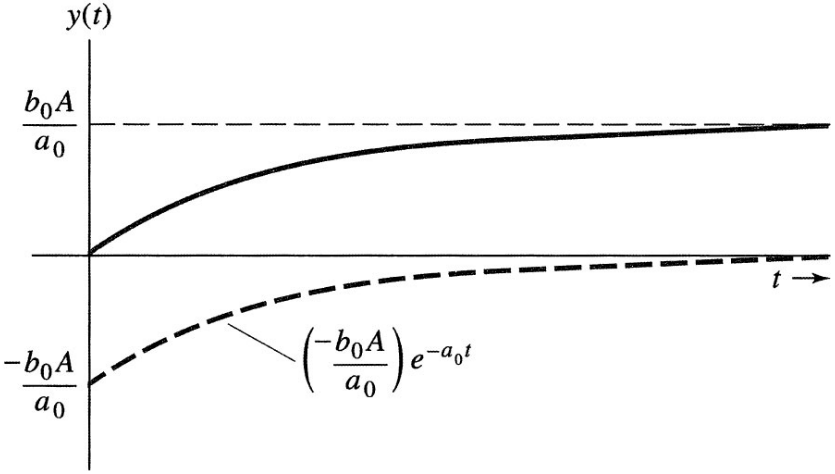
$$\frac{dy}{dt}+a_0 y=b_0 r$$

$$Y(s)=\frac{b_0}{s+a_0}R(s)+\frac{1}{s+a_0}y(0)$$

$$let \qquad r(t)=Au(t)$$

$$Y(s)=\frac{A\frac{b_0}{a_0}}{s}+\frac{-A\frac{b_0}{a_0}}{s+a_0}+\frac{1}{s+a_0}y(0)$$

$$y(t)=\frac{Ab_0}{a_0}u(t)-\frac{Ab_0}{a_0}e^{-a_0 t}+y(0)e^{-a_0 t}$$



Second order

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = b_1\frac{dr}{dt} + b_0 r$$

$$Y(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} R(s) + \frac{sy'(0) + (a_1 + 1)y(0)}{s^2 + a_1 s + a_0}$$

Three cases :

(a) Two characteristic roots are real and distinct.

(b) Two characteristic roots are equal.

(c) Two characteristic roots are complex numbers.

Second order

Two characteristic roots are real and distinct.

$$let \qquad y'(0) = y(0) = 0 \qquad r(t) = u(t)$$

$$Y(s) = \frac{k_1}{s - s_1} + \frac{k_2}{s - s_2} + \frac{k_3}{s}$$

$$y(t) = (k_1 e^{s_1 t} + k_2 e^{s_2 t} + k_3) u(t)$$

Second order

Two characteristic roots are equal

$$let \qquad y'(0) = y(0) = 0 \qquad r(t) = u(t)$$

$$Y(s) = \frac{k_1}{(s - s_1)^2} + \frac{k_2}{s - s_1} + \frac{k_3}{s}$$

$$y(t) = (k_1 e^{s_1 t} + k_2 t e^{s_1 t} + k_3) u(t)$$

Second order

Two characteristic roots are complex numbers

$$let \qquad y'(0) = y(0) = 0 \qquad r(t) = u(t)$$

$$Y(s) = \frac{k_1}{(s+\sigma)^2 + \omega^2} R(s)$$

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} R(s)$$

$$\sigma = \xi\omega_n$$

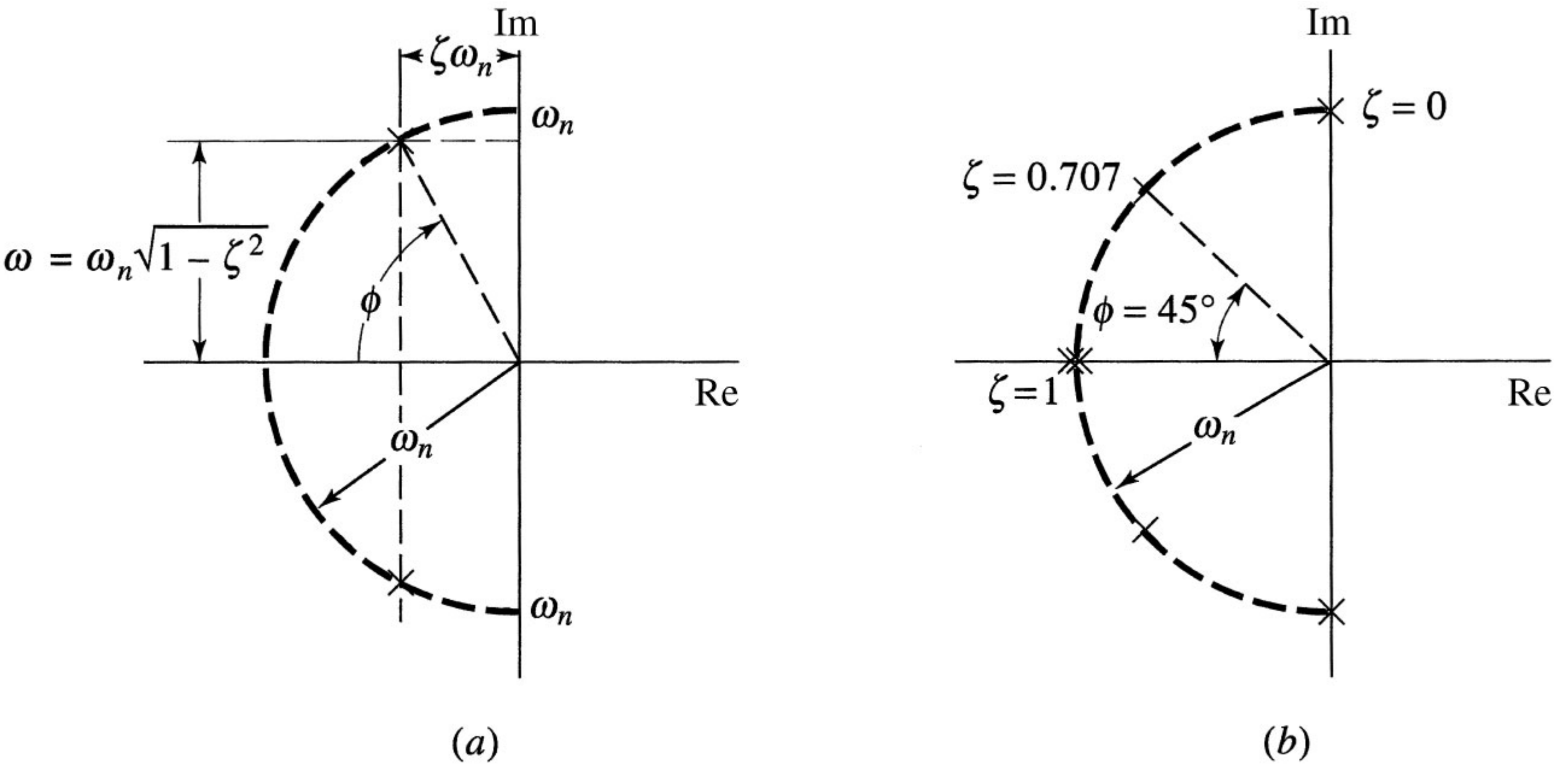
$$\omega = \omega_n\sqrt{1-\xi^2}$$

Damping ratio

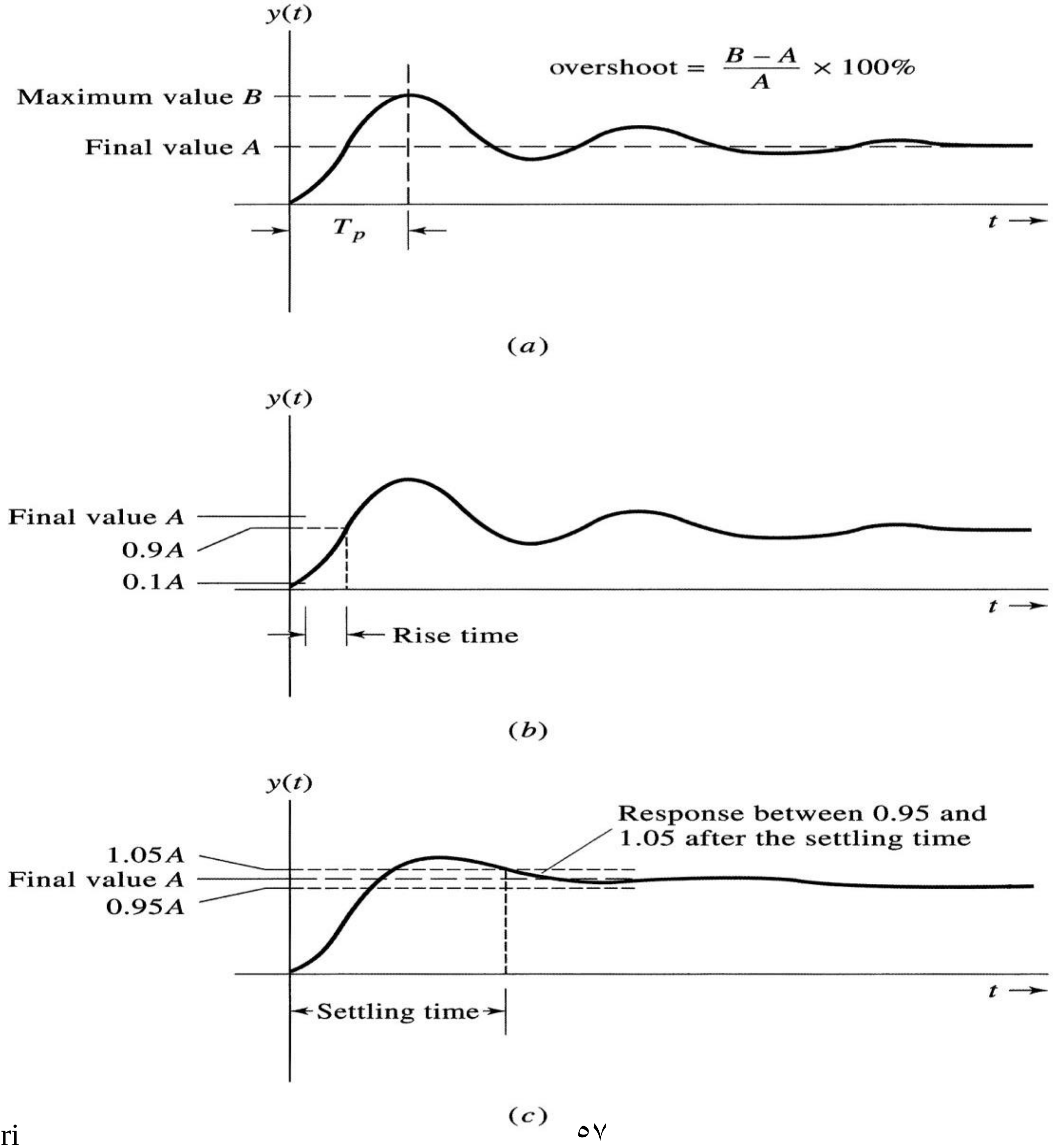
Undamped natural frequency

$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_n\sqrt{1-\xi^2}\,t + \cos^{-1}\xi)$$

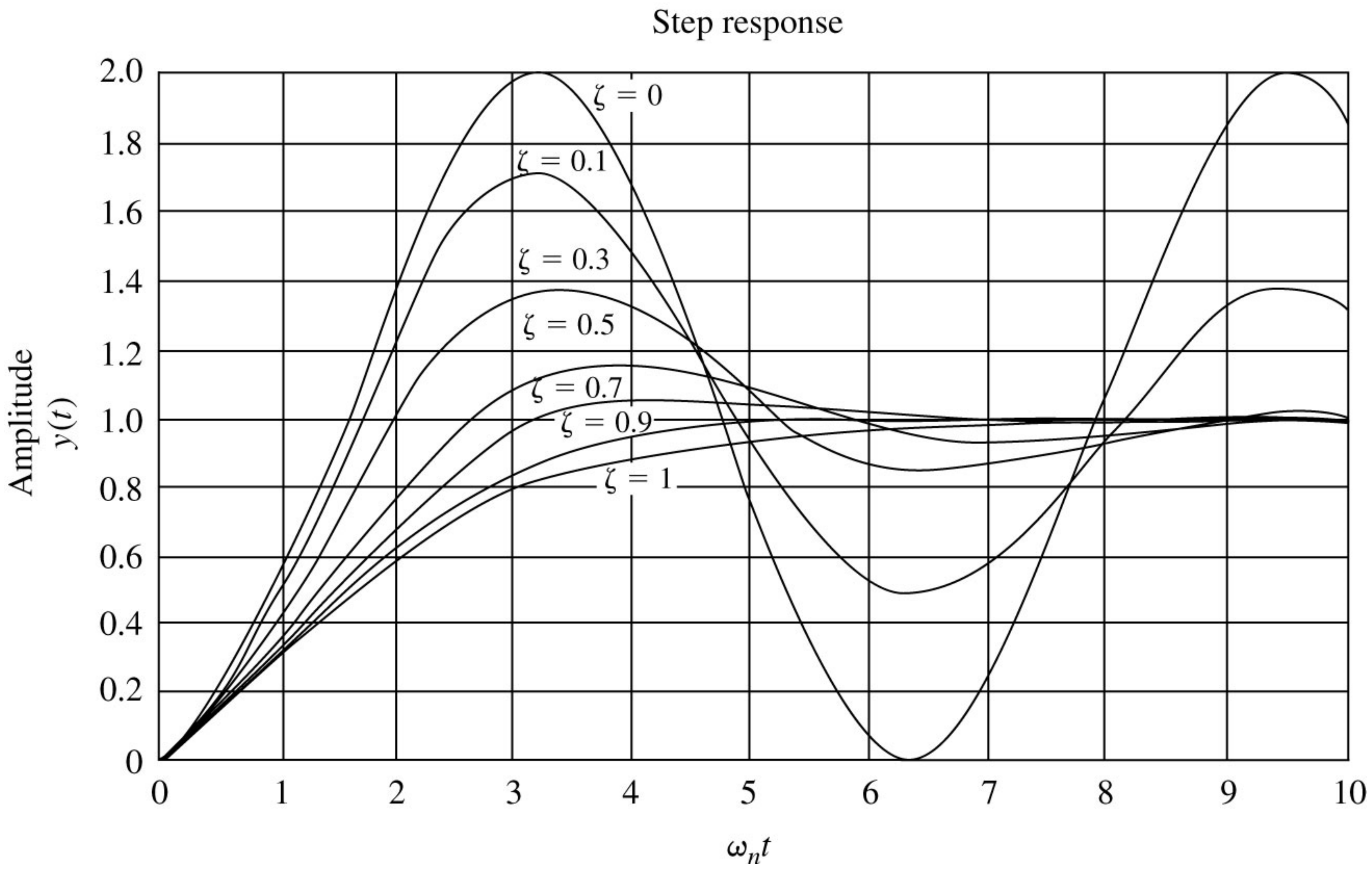
Second order



Second order



Second order

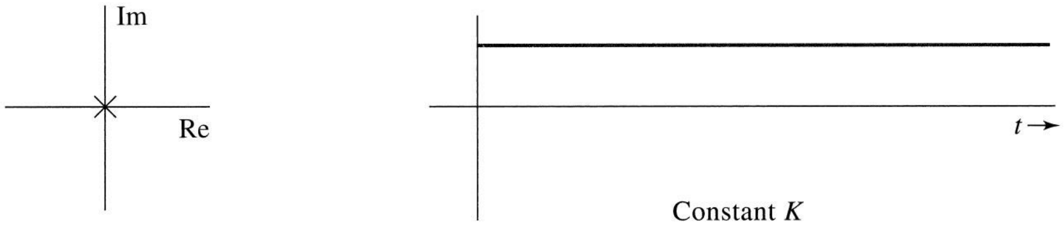


Higher-Order System

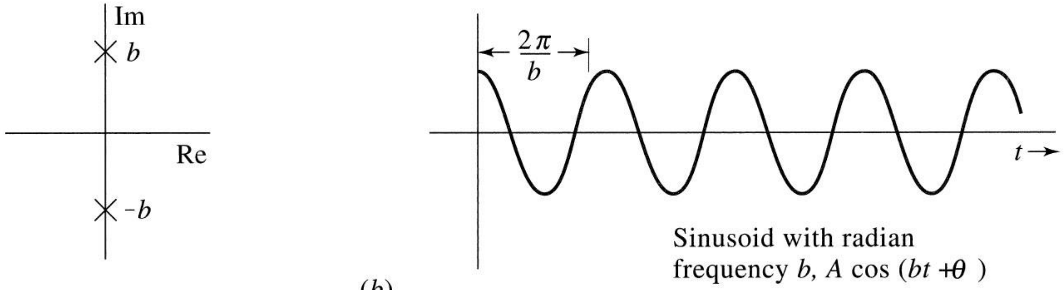
$$Y(s)=\frac{-8s^2+5}{s^4+9s^3+37s^2+81s+52}$$

$$Y(s)=\underbrace{\frac{k_1}{s+1}}_{\text{Dominant root}}+\underbrace{\frac{k_2}{s+4}}_{\text{nondominant root}}+\frac{k_3s+k_4}{s^2+4s+13}$$

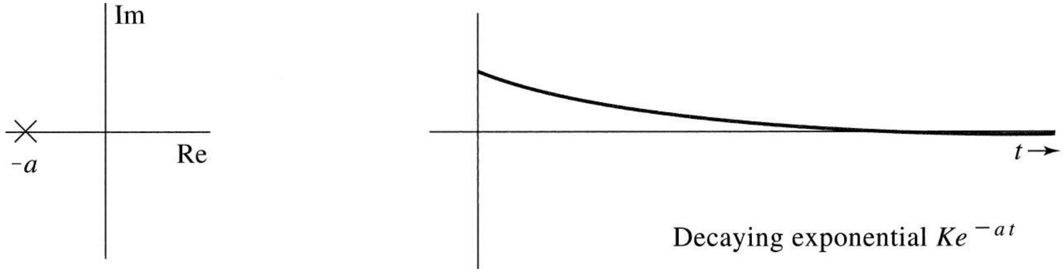
Denominator Root Locations **Time Function**



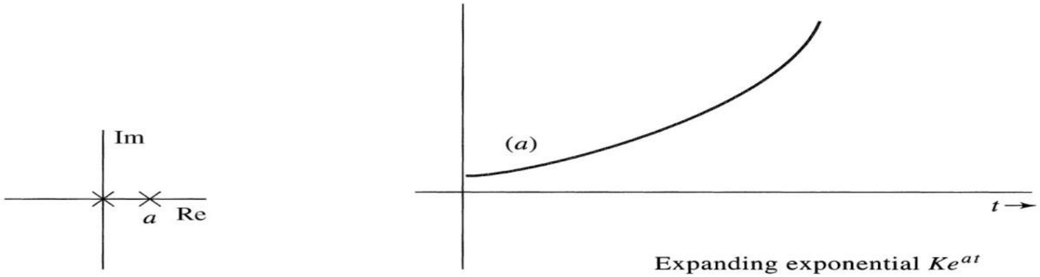
(a)



(b)

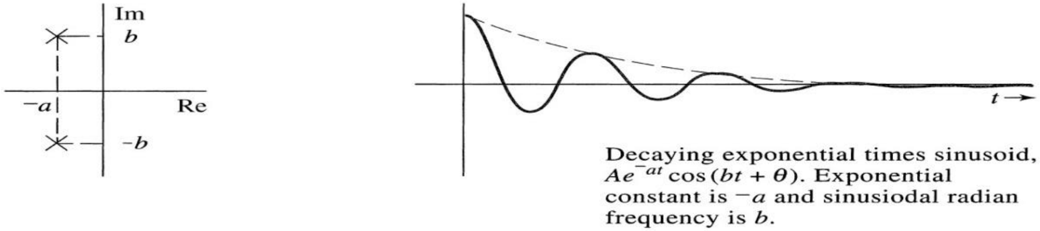


(c)



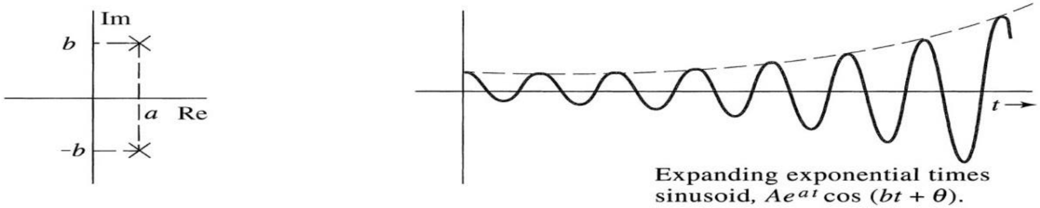
Expanding exponential Ke^{at}

(d)



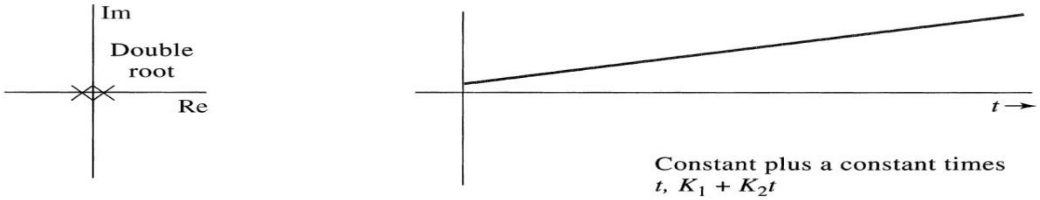
Decaying exponential times sinusoid, $Ae^{-at}\cos(bt+\theta)$. Exponential constant is $-a$ and sinusiodal radian frequency is b.

(e)



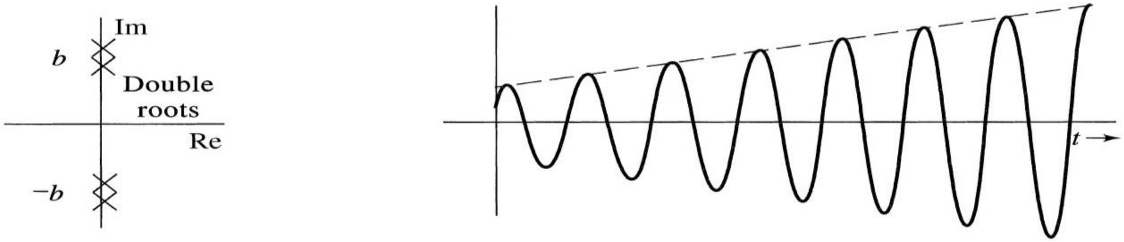
Expanding exponential times sinusoid, $Ae^{at}\cos(bt+\theta)$.

(f)



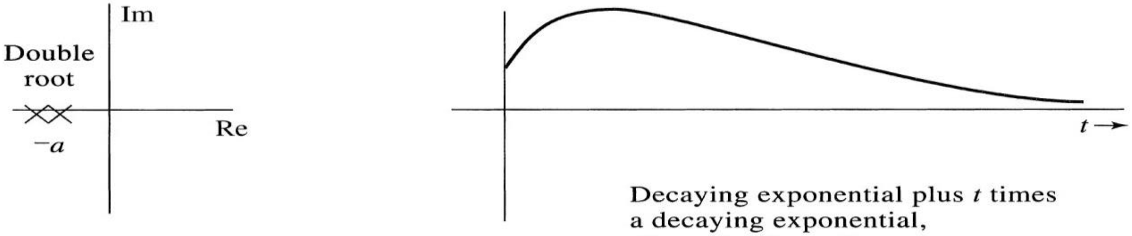
Constant plus a constant times t, $K_1 + K_2t$

(g)



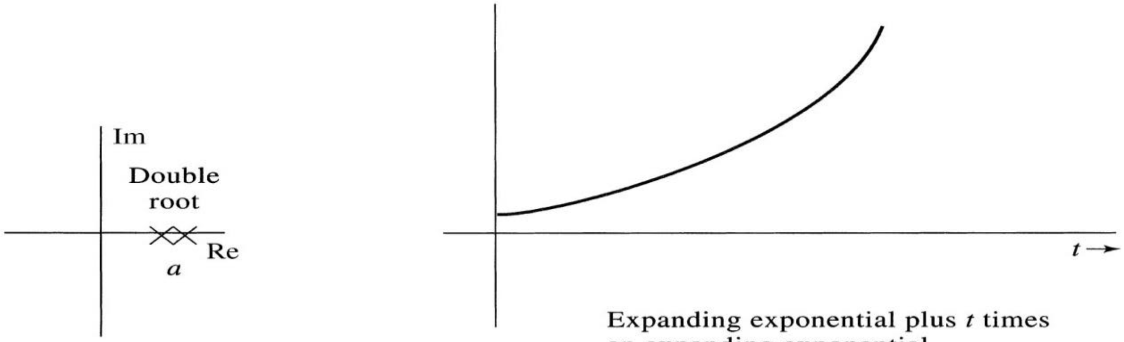
Sinusoid plus t times a sinusiod,
$A_1 \cos(bt + \theta_1) + A_2 t \cos(bt + \theta_2)$

(*h*)



Decaying exponential plus t times
a decaying exponential,
$K_1 e^{-at} + K_2 t e^{-at}$

(*i*)



Expanding exponential plus t times
an expanding exponential,
$K_1 e^{at} + K_2 t e^{at}$

(*j*)

MENG366

Routh's Stability Criterion

Part II

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A necessary condition for Routh Stability

- A *necessary condition for stability* of the system is that all of the roots of its characteristic equation have negative real parts, which in turn requires that ***all the coefficients be positive.***

A necessary (but not sufficient) condition for stability is that ***all*** **the coefficients of the polynomial characteristic equation be positive.**

A necessary and sufficient condition for Stability

- Routh's formulation requires the computation of a triangular array that is a function of the coefficients of the polynomial characteristic equation.

A **system is stable** if and only if ***all*** **the elements of the first column** of the Routh array are **positive**

Characteristic Equation

- Consider an nth-order system whose the characteristic equation (*which is also the denominator of the transfer function*) is:

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_0 s^0$$

Stability testing

Properties of the polynomial coefficients :

- Differing algebraic signs

$$7s^6 + 5s^4 - 3s^3 - 2s^2 + s + 10$$

At least one RHP root

- Zero-valued coefficients

$$s^6 + 3s^5 + 2s^4 + 8s^2 + 3s + 17$$

Has imaginary axis roots or RHP roots or both

- All of the same algebraic sign, non zero

$$8s^5 + 6s^4 + 3s^3 + 2s^2 + 7s + 10$$

No direct information

Method for determining the Routh array

- Consider the characteristic equation:

$$a(s) = 1 \times s^{n} + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \ldots + a_{n-1} s^{1} + a_0 s^{0}$$

- First arrange the coefficients of the characteristic equation in two rows, beginning with the first and second coefficients and followed by the even-numbered and odd-numbered coefficients:

$$\begin{array}{lcccc} s^{n}: & 1 & a_2 & a_4 & \ldots \\ s^{n-1}: & a_1 & a_3 & a_5 & \ldots \end{array}$$

Routh array: method (cont'd)

- Then add subsequent rows to complete the **Routh array:**

?!

$$
\begin{array}{llllll}
s^n & : & 1 & a_2 & a_4 & \ldots \\
s^{n-1} & : & a_1 & a_3 & a_5 & \ldots \\
s^{n-2} & : & b_1 & b_2 & b_3 & \ldots \\
s^{n-3} & : & c_1 & c_2 & c_3 & \ldots \\
\vdots & & \vdots & \vdots & \vdots & \\
s^2 & : & * & * & & \\
s^1 & : & * & & & \\
s^0 & : & * & & &
\end{array}
$$

Routh array: method (cont'd)

- Compute elements for the 3rd row:

$$b_1 = -\frac{1\times a_3 - a_2 a_1}{a_1},$$

$$b_2 = -\frac{1\times a_5 - a_4 a_1}{a_1},$$

$$b_3 = -\frac{1\times a_7 - a_6 a_1}{a_1}$$

$\ldots$

$$\begin{array}{rcccc}
s^n: & 1 & a_2 & a_4 & \ldots \\
s^{n-1}: & a_1 & a_3 & a_5 & \ldots \\
s^{n-2}: & b_1 & b_2 & b_3 & \ldots \\
s^{n-3}: & c_1 & c_2 & c_3 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \\
s^2: & * & * & & \\
s^1: & * & & & \\
s^0: & * & & &
\end{array}$$

Routh array: method (cont'd)

- Compute elements for the 4th row:

$$c_1 = -\frac{a_1 \times b_2 - a_3 b_1}{b_1},$$

$$c_2 = -\frac{a_1 \times b_3 - a_5 b_1}{b_1},$$

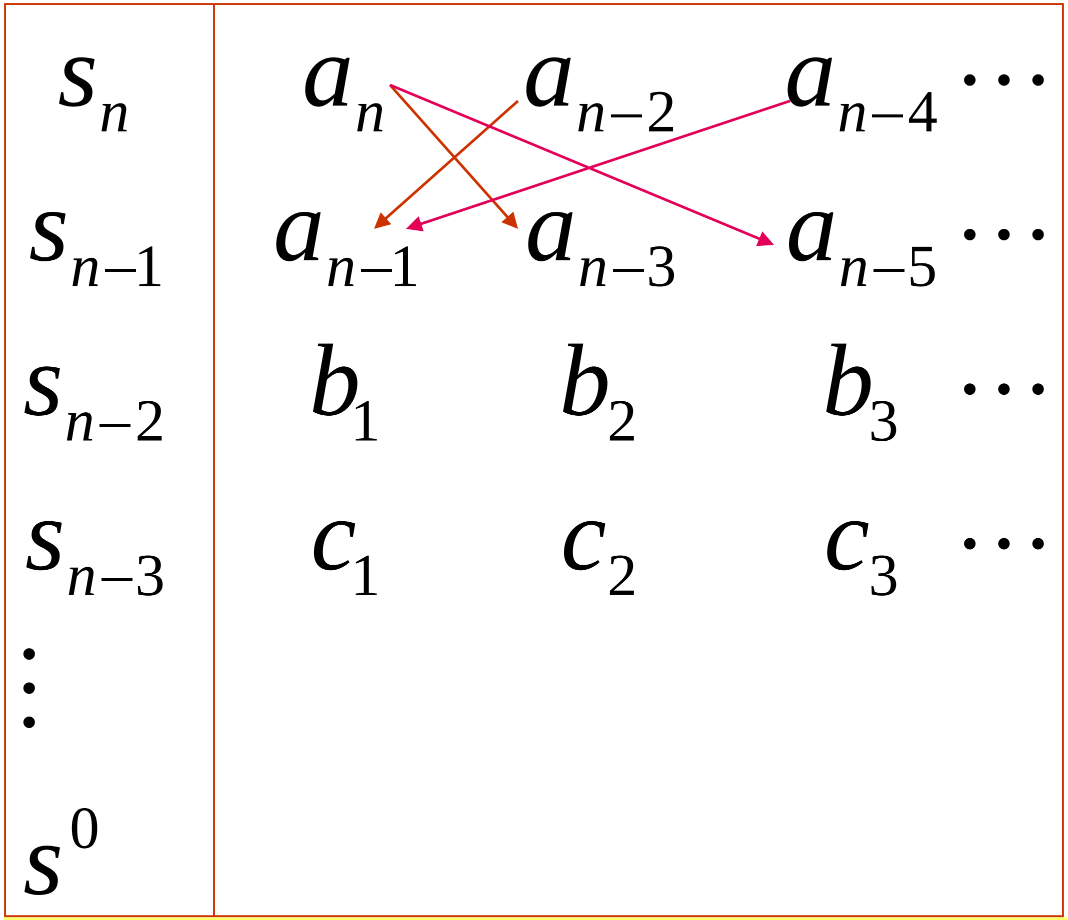
$$c_3 = -\frac{a_1 \times b_4 - a_7 b_1}{b_1}$$

$$\ldots$$

$$\begin{array}{llllll}
s^n : & 1 & a_2 & a_4 & \ldots \\
s^{n-1} : & a_1 & a_3 & a_5 & \ldots \\
s^{n-2} : & b_1 & b_2 & b_3 & \ldots \\
s^{n-3} : & c_1 & c_2 & c_3 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \\
s^2 : & * & * & & \\
s^1 : & * & & & \\
s^0 : & * & & &
\end{array}$$

Routh-Hurwitz testing

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$



$$\begin{array}{c|cccc} s_n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s_{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s_{n-2} & b_1 & b_2 & b_3 & \cdots \\ s_{n-3} & c_1 & c_2 & c_3 & \cdots \\ \vdots & & & & \\ s^0 & & & & \end{array}$$

$$b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1}$$

The number of RHP roots of *P(s)* is the number of algebra sign changes in the elements of the left column of the array.

Example 1:

Given the characteristic equation,

$$a(s) = s^6 + 4s^5 + 3s^4 \boxed{-2}s^3 + s^2 + 4s + 4$$

is the system described by this characteristic equation stable?

Answer:

- One coefficient (-2) is negative.
- Therefore, the system **does not** satisfy the necessary condition for stability.

Example 2:

Given the characteristic equation,

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

is the system described by this characteristic equation stable?

Answer:

- All the coefficients are positive and nonzero.
- Therefore, the system **satisfies** the necessary condition for stability.
- We should determine whether any of the coefficients of the first column of the Routh array are negative.

Example 2 (cont'd): Routh array

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$$\begin{array}{llllll}
s^6 & : & 1 & 3 & 1 & 4 \\
s^5 & : & 4 & 2 & 4 & 0 \\
s^4 & : & ? & ? & ? & \\
s^3 & : & ? & ? & ? & \\
s^2 & : & ? & ? & & \\
s^1 & : & ? & ? & & \\
s^0 & : & ? & & &
\end{array}$$

Example 2 (cont'd): Routh array

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$$\begin{array}{llllll}
s^6: & 1 & 3 & 1 & 4 \\
s^5: & 4 & 2 & 4 & 0 \\
s^4: & 5/2 & 0 & 4 & \\
s^3: & ? & ? & ? & \\
s^2: & ? & ? & & \\
s^1: & ? & ? & & \\
s^0: & ? & & &
\end{array}$$

$$-\frac{1\times 2 - 3\times 4}{4} = -\frac{-10}{4} = \frac{5}{2}$$

$$-\frac{1\times 4 - 1\times 4}{4} = -\frac{0}{4} = 0$$

$$-\frac{1\times 0 - 4\times 4}{4} = -\frac{-16}{4} = 4$$

Example 2 (cont’d): Routh array

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$$\begin{array}{lcccc}
s^6: & 1 & 3 & 1 & 4 \\
s^5: & 4 & 2 & 4 & 0 \\
s^4: & 5/2 & 0 & 4 & \\
s^3: & 2 & -12/5 & 0 & \\
s^2: & ? & ? & & \\
s^1: & ? & ? & & \\
s^0: & ? & & &
\end{array}$$

$$-\frac{4 \times 0 - 2 \times 5/2}{5/2} = -(-2) = 2$$

$$-\frac{4 \times 4 - 4 \times 5/2}{5/2} = -\frac{32 - 20}{5} = \frac{-12}{5}$$

Example 2 (cont'd): Routh array

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$$\begin{array}{lllll}
s^6: & 1 & 3 & 1 & 4 \\
s^5: & 4 & 2 & 4 & 0 \\
s^4: & 5/2 & 0 & 4 & \\
s^3: & 2 & -12/5 & 0 & \\
s^2: & 3 & 4 & & \\
s^1: & \boxed{-76/15} & 0 & & \\
s^0: & 4 & & &
\end{array}$$

The elements of the 1st column **are not all** positive:

the system is unstable

Example 1

$$P(s) = 2s^4 + 3s^3 + 5s^2 + 2s + 6$$

s^4	2	5	6
s^3	3	2	
s^2	$\frac{15-4}{3} = \frac{11}{3}$	$\frac{18-0}{3} = 6$	
s^1	$\frac{-32}{11}$		
s^0	6		

Two roots in the RHP

Example 2

$$P(s) = s^4 + 2s^3 + 3s^2 + 4s + 1$$

$$\begin{array}{llll}
s^4 & 1 & 3 & 1 \\
s^3 & 2 & 4 & \\
s^2 & 1 & 1 & \\
s^1 & 2 & & \\
s^0 & 1 & &
\end{array}$$

no root in the RHP

POP Quiz: Study the stability of the following system:

$$P(s) = 3s^4 + 6s^3 + 2s^2 + 4s + 5$$

Example 3

$$P(s) = 3s^4 + 6s^3 + 2s^2 + 4s + 5$$

$$\begin{array}{llll}
s^4 & 3 & 2 & 5 \\
s^3 & 6 & 4 & \\
s^2 & \varepsilon & 5 & \\
s^1 & \dfrac{4\varepsilon - 30}{\varepsilon} & & \\
s^0 & 5 & &
\end{array}$$

Example 4

$$P(s) = s^5 + 2s^4 + 8s^3 + 11s^2 + 16s + 12$$

Example 4

$$P(s) = s^5 + 2s^4 + 8s^3 + 11s^2 + 16s + 12$$

$$\begin{array}{llll}
s^5 & 1 & 8 & 16 \\
s^4 & 2 & 11 & 12 \\
s^3 & 2.5 & 10 & \\
s^2 & 3 & 12 & \\
s^1 & 0 & 0 & \\
s^0 & & &
\end{array}$$

factor

$$3s^2 + 12$$

Example 4 (Cont.)

$$P(s) = s^5 + 2s^4 + 8s^3 + 11s^2 + 16s + 12$$

$$\begin{array}{l|c|cc}
s^5 & 1 & 8 & 16 \\
s^4 & 2 & 11 & 12 \\
s^3 & 2.5 & 10 & \\
s^2 & 3 & 12 & \\
s^1 & 6 & & \\
s^0 & 12 & &
\end{array}$$

$$\frac{d}{ds}(3s^2 + 12) = 6s$$

no root in the RHP

Special Case 1 - Another Method

As the characteristics equation is

$$s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$$

Let, *s=1/d*

$$\Rightarrow \left(\frac{1}{d}\right)^n + a_{n-1}\left(\frac{1}{d}\right)^{n-1} + \ldots + a_1\left(\frac{1}{d}\right) + a_0 = 0$$

Then, *d* are reciprocal of the roots *s*

$$\Rightarrow \left(\frac{1}{d}\right)^n \left[1 + a_{n-1}d + \ldots + a_1 d^{n-1} + a_0 d^n\right] = 0$$

Example on Routh Stability

Determine the stability of the closed-loop transfer function

$$T(s)=\frac{10}{s^5+2s^4+3s^3+6s^2+5s+3}$$

➡ **Characteristics Eq^{n}:**

$$\boxed{3d^5+5d^4+6d^3+3d^2+2d+1=0}$$

Example on Routh Stability

d^5	3	6	2
d^4	5	3	1
d^3	4.2	1.4	
d^2	1.33	1	
d^1	-1.75		
d^0	1		

Two sign changes → two roots with positive real parts

No Zeros in first column

Routh-Hurwitz Stability Criterion

Special Case 2

- An **all zero row** in the Routh array which corresponds to pairs of roots with opposite signs.

- Remedy:
  - form an **auxiliary polynomial** from the coefficients in the row above.
  - Replace the zero coefficients from the coefficients of the **differentiated auxiliary polynomial**.
  - If there is no sign change, the roots of the auxiliary equation define the roots of the system on the imaginary axis.

Meaning of Entire Row of Zeros

Special Case 2 - Example

For purely even or odd polynomials, as

$$s^4 + 5s^2 + 6 = 0$$

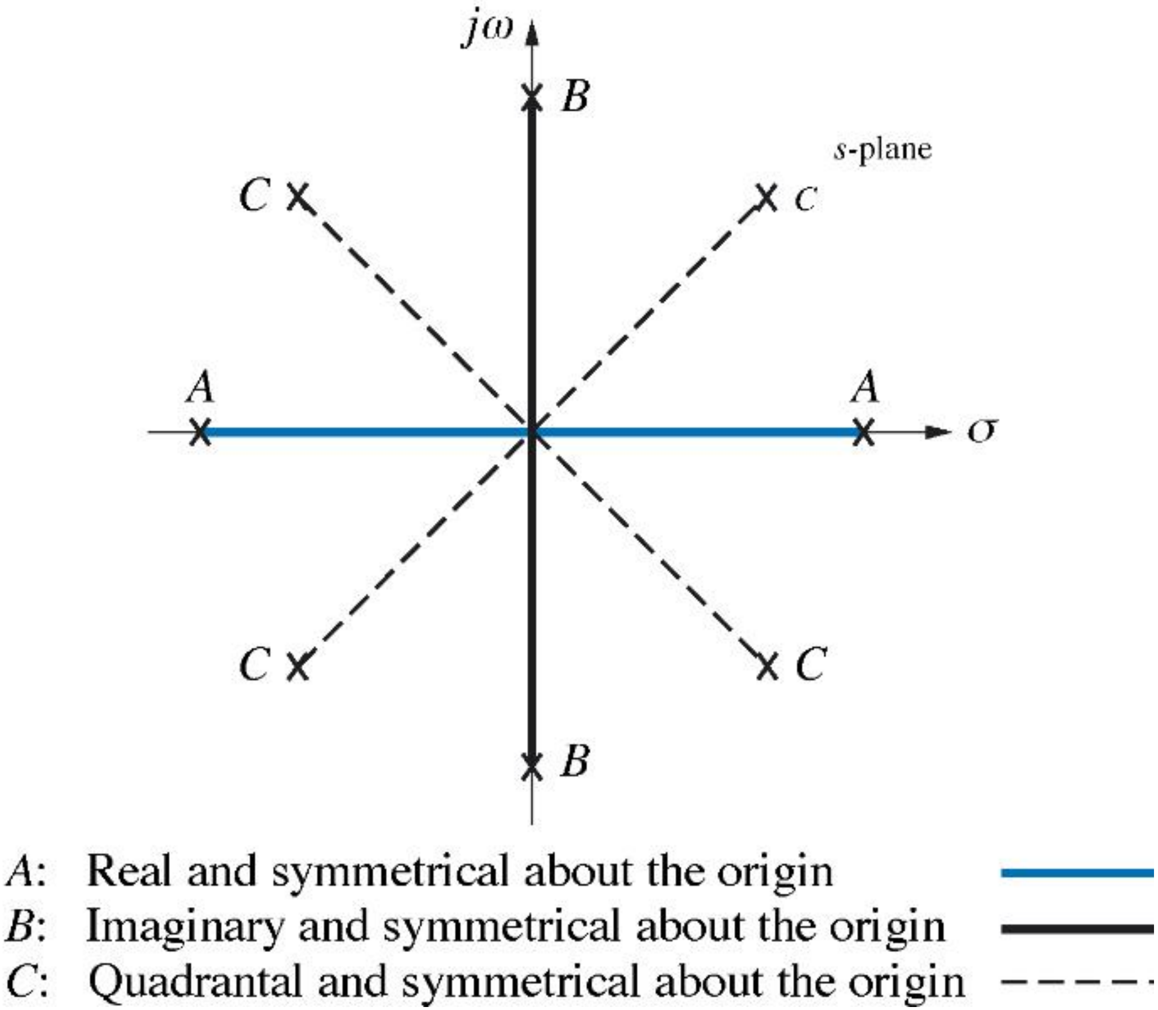
Row of zeros as:

s^4	1	5	6	0
s^3	0	0	0	0
s^2				
s^1				
s^0				

s^4	1	5	6	0
s^3	4	10	0	
s^2	2.5	6	0	
s^1	0.4			
s^0	6			

Meaning of Entire Row of Zeros

```
>> p=[1 0 5 0  6];
>> roots(p)

ans =


        0 + 1.7321i
        0 - 1.7321i
        0 + 1.4142i
        0 - 1.4142i
```



Meaning of Entire Row of Zeros

```
>> Routh_H([1  0  5  0 6])
-------------------------------------------------------------------------
                         RESULTS
-------------------------------------------------------------------------
  Not satisfied with the necessary condition.
  The system is critically stable and has 4 pole(s) in the imaginary axis .
  There were rows of zeroes in the array in the row(s) 2.
-------------------------------------------------------------------------


 ROUTH-HURWITZ ARRAY

  s^4 |     1      |     5      |     6      |
  s^3 |     4      |     10     |
  s^2 |    2.5     |     6      |
  s^1 |    0.4     |
  s^0 |     6      |
```

Routh-Hurwitz Stability Criterion

Parameter Range Test

- The Routh-Hurwitz stability criterion may be used to find the **range of a parameter** for which the closed-loop systems is stable.

- Leave the parameter as an unknown coefficient in the characteristic polynomial, form the Routh array, check the range of the parameter such that the first column does not change sign.

Routh-Hurwitz Stability Criterion

Parameter Range Example

$$Q(s) = s^4 + 6s^3 + 11s^2 + 6s + K$$

s^4	1	11	K	0
s^3	6	6	0	0
s^2	10	K	0	
s^1	c_1	0		
s^0	d_1			

$$c_1 = \frac{60 - 6K}{10} \qquad d_1 = K$$

Stability requires:

$$K > 0$$

$$60 - 6K > 0 \Rightarrow K < 10$$

$$\boxed{\therefore \quad 0 < K < 10}$$

Routh-Hurwitz Stability Criterion

Determine the stability of the closed-loop transfer function

$$T(s)=\frac{10}{s^5+7s^4+6s^3+42s^2+8s+56}$$

Routh-Hurwitz Stability Criterion

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

$$P(s) = s^4 + 6s^2 + 8$$

$$\Downarrow$$

$$\frac{dP(s)}{ds} = 3s^3 + 12s + 0 \Rightarrow s^3 + 4s + 0$$

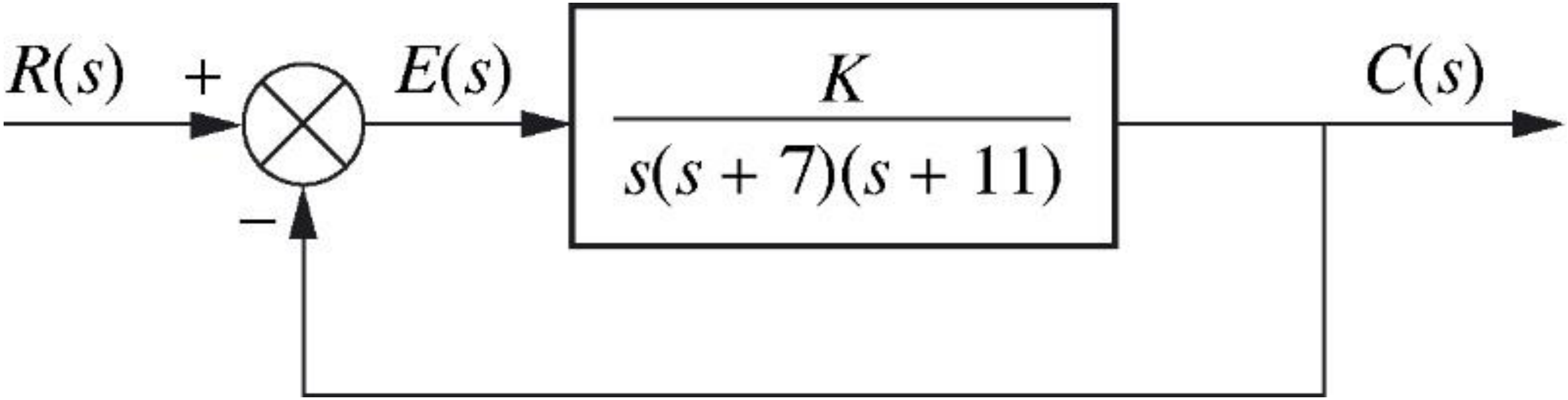
Marginally Stable System

Routh-Hurwitz Stability Criterion

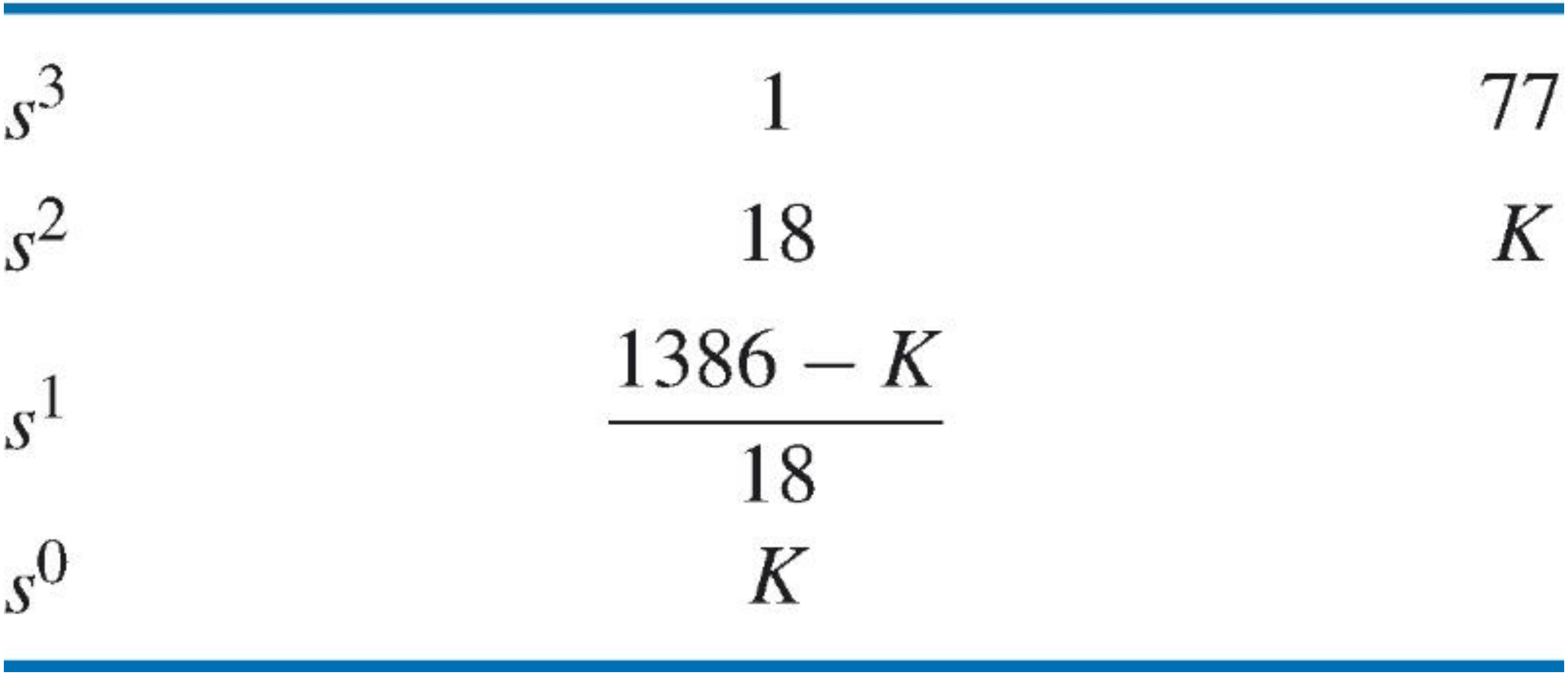
```
>> Routh_H([1 7 6 42 8 56])
--------------------------------------------------------------------------------
                         RESULTS
--------------------------------------------------------------------------------
   The system is critically stable and has 4 pole(s) in the imaginary axis .
   There were rows of zeroes in the array in the row(s) 3.
--------------------------------------------------------------------------------


 ROUTH-HURWITZ ARRAY

   s^5 |     1       |     6     |     8     |
   s^4 |     7       |     42    |     56    |
   s^3 |     28      |     84    |
   s^2 |     21      |     56    |
   s^1 |   9.3333    |
   s^0 |     56      |
```

Routh-Hurwitz Stability Criterion

Find the range of *K*, for which the following system will be stable, marginally stable, & unstable. Assume *K*>0.



Routh-Hurwitz Stability Criterion



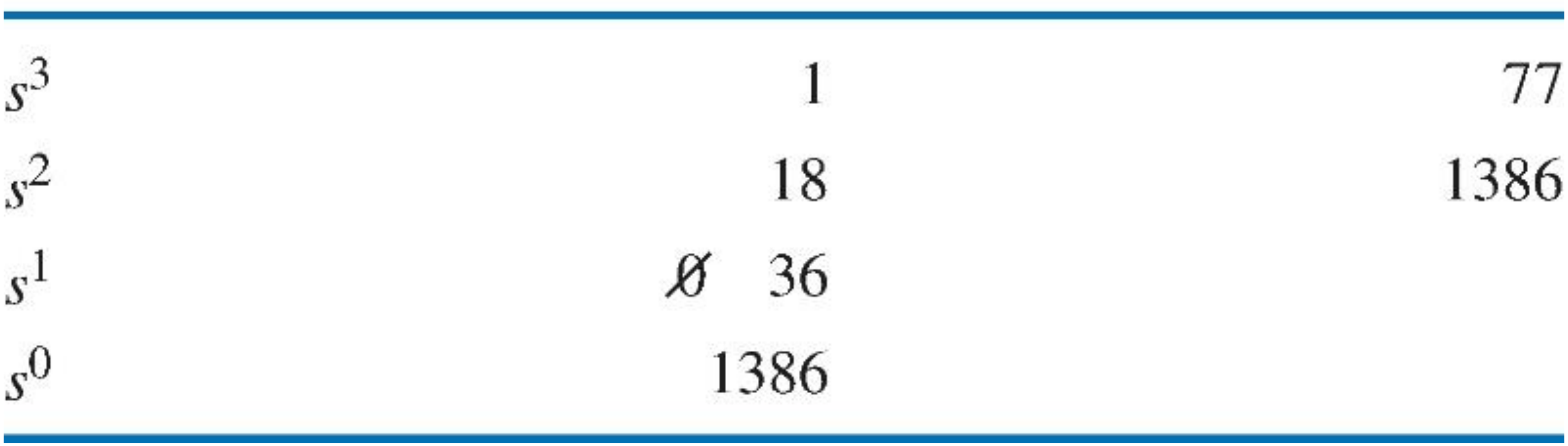
s^3	1	77
s^2	18	K
s^1	$\dfrac{1386-K}{18}$	
s^0	K	

$K<1386$ → Stable

$K>1386$ → Unstable

Routh-Hurwitz Stability Criterion

K=1386



s^3	1	77
s^2	18	1386
s^1	$\cancel{0}$ 36	
s^0	1386	

Since there is no sign change → system is marginally stable

Routh-Hurwitz Stability Criterion

K=1386

➡ $$\frac{C}{R}=\frac{1386}{s^3+18s^2+77s+1386}$$

```
>> p=[1 18 77 1386];
>> roots(p)

ans =


 -18.0000
        0 + 8.7750i
        0  - 8.7750i
```

Since there is no sign change → system is marginally stable

Routh-Hurwitz Stability Criterion

```
>> Routh_H([1  18 77 1386])
---------------------------------------------------------------------------------------------------
                              RESULTS
---------------------------------------------------------------------------------------------------
   The system is critically stable and has 2 pole(s) in the imaginary axis .
   There were rows of zeroes in the array in the row(s) 3.
---------------------------------------------------------------------------------------------------


 ROUTH-HURWITZ ARRAY

   s^3  |      1       |     77     |
   s^2  |      18      |    1386    |
   s^1  |      36      |
   s^0  |     1386     |
```

POP. Quiz

$$T(s) = \frac{K(s+6)}{s^3 + 4s^2 + (K+3)s + 6K}$$

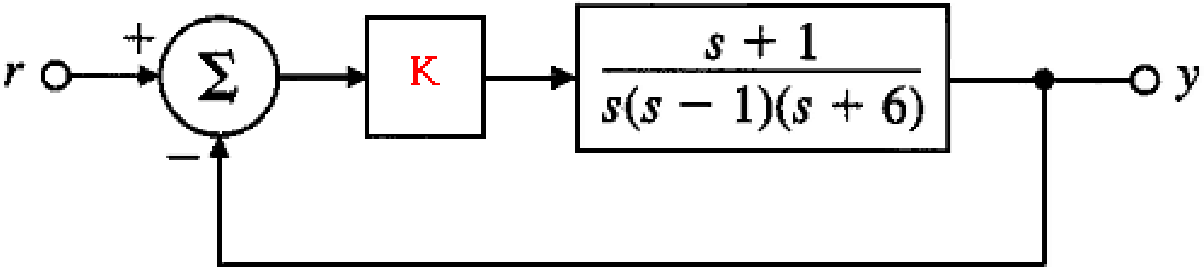
$$T(s) = \frac{K(s+6)}{s^3 + 4s^2 + (K+3)s + 6K}$$

s^3	1	$3 + K$
s^2	4	$6K$
s^1	$3 - \frac{1}{2}K$	0
s^0	$6K$	0

Stable for $0 < K < 6$

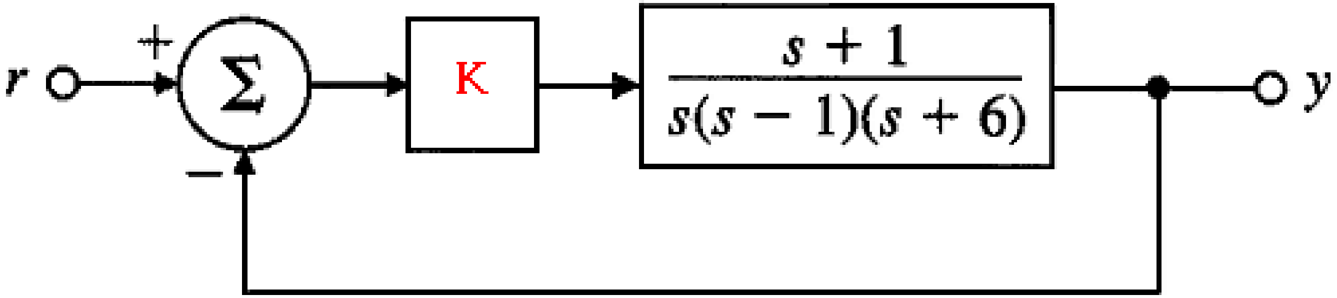
Example 5: Stability versus Parameter Range

Consider a feedback system such as:



The stability properties of this system are a function of the proportional feedback gain *K*. Determine the range of *K* over which the system is stable.

Example 5 (cont'd)



- The characteristic equation for the system is given by:

$$1+K\frac{s+1}{s(s-1)(s+6)}=0$$

Example 5 (cont'd)

$$1+K\frac{s+1}{s(s-1)(s+6)}=0$$

- Expressing the characteristic equation in polynomial form, we obtain:

$$s^3+5s^2+(K-6)s+K=0$$

Example 5 (cont'd)

$$a(s) = s^3 + 5s^2 + (K-6)s + K$$

- The corresponding Routh array is:

$$\begin{array}{lcc} s^3: & 1 & K-6 \\ s^2: & 5 & K \\ s^1: & (4K-30)/5 & \\ s^0: & K & \end{array}$$

- Therefore, the **system is stable if and only if**

$$\frac{4K-30}{5} > 0 \text{ and } K > 0$$

$$\Rightarrow \quad K > 7.5 \text{ and } K > 0$$

$$\Rightarrow \boxed{K > 7.5}$$

Example 5 (cont'd)

$$a(s) = s^3 + 5s^2 + (K - 6)s + K$$

- Solving for the roots using MATLAB gives:

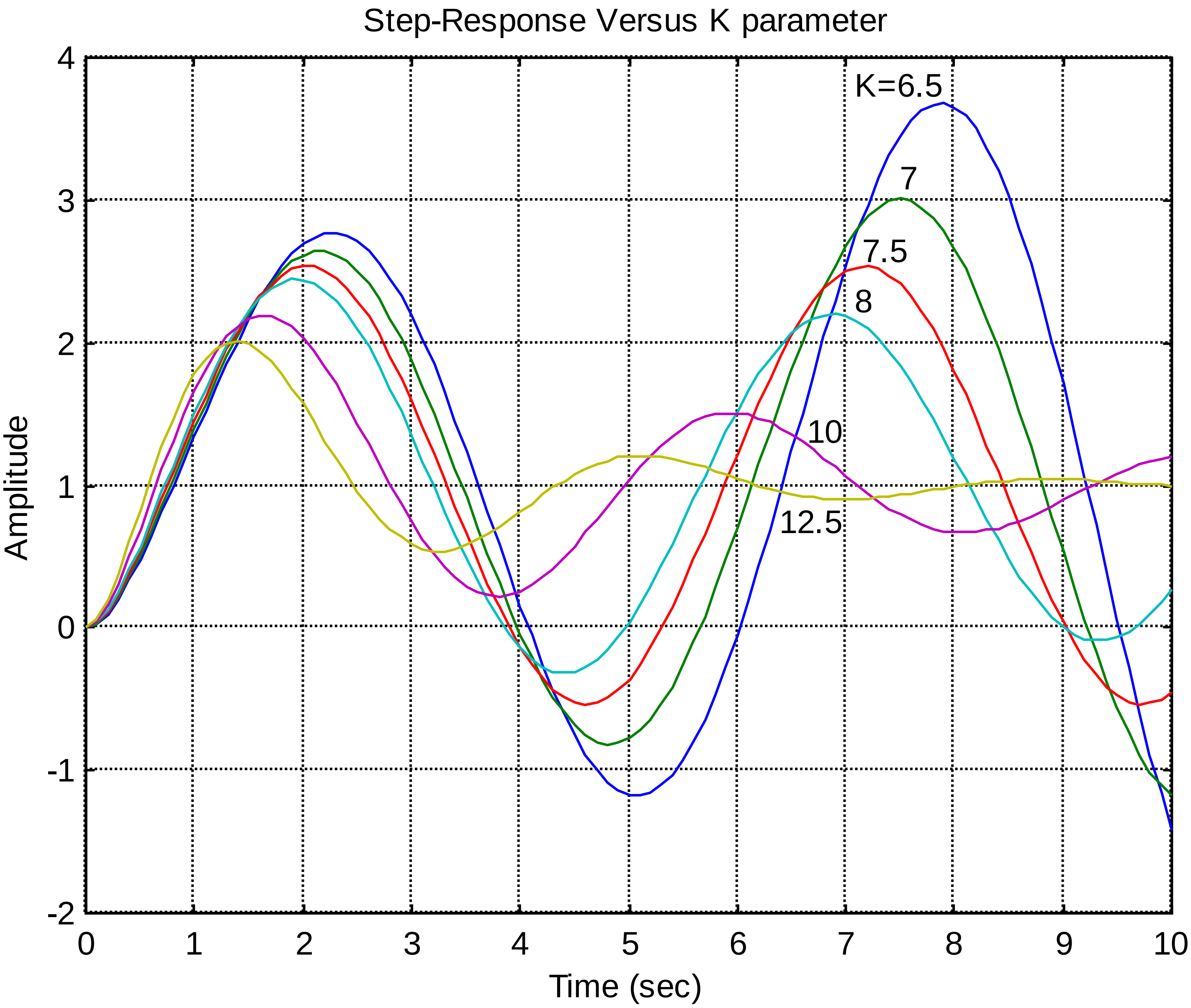
-5 and ±1.22j for *K*=7.5

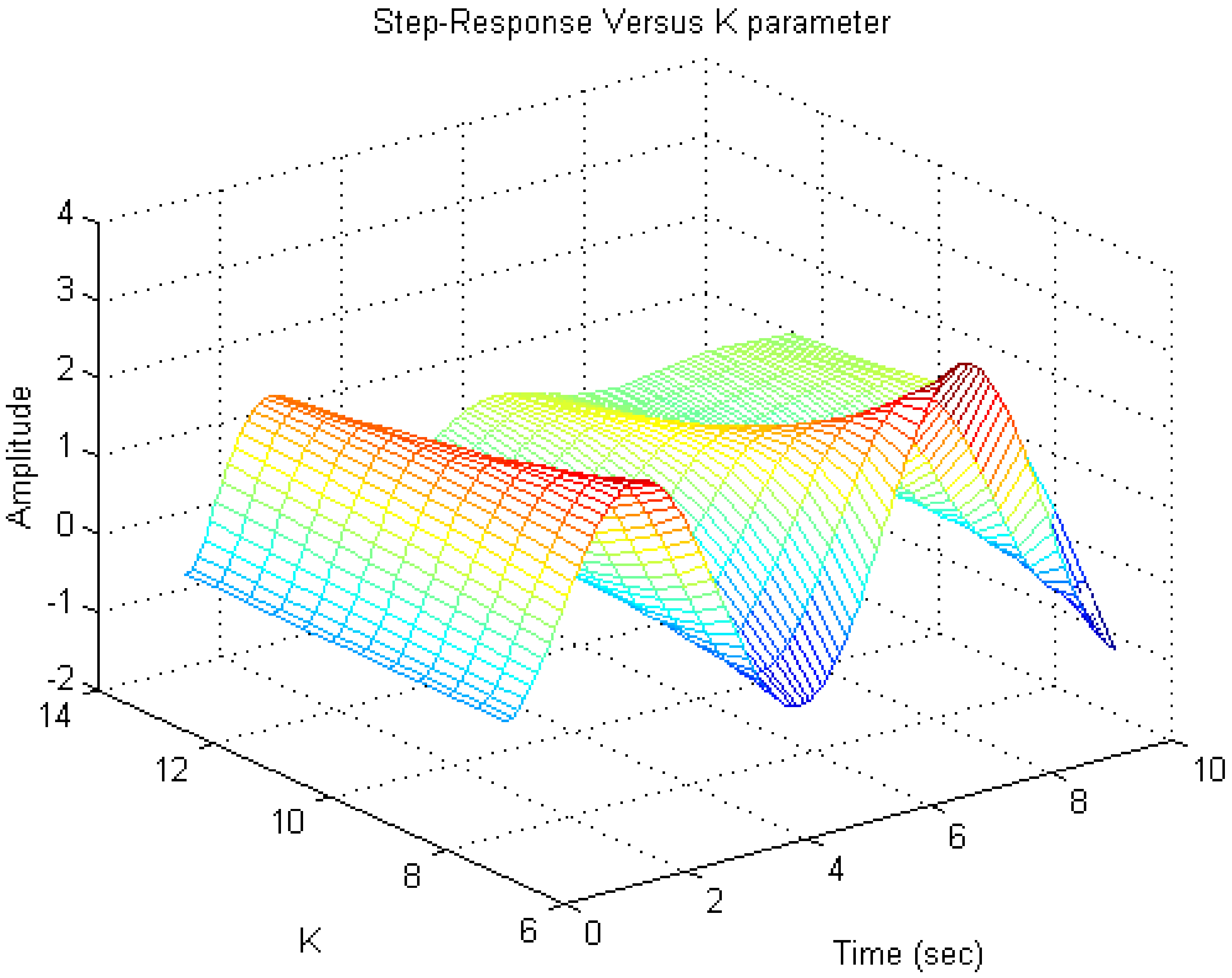
=> The system is unstable (or critically stable) for *K*=7.5

-4.06 and –0.47 ±1.7j for *K*=13

-1.90 and –1.54 ±3.27j for *K*=25

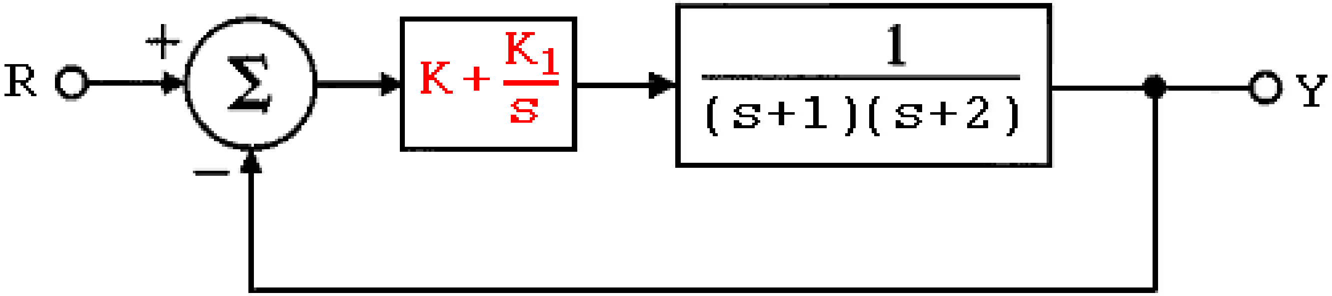
=> The system is stable for both *K*=13 and *K*=25


Step-Response Versus K parameter
K=6.5
7
7.5
8
10
12.5
Amplitude
4
3
2
1
0
-1
-2
0
1
2
3
4
5
6
7
8
9
10
Time (sec)


Step-Response Versus K parameter

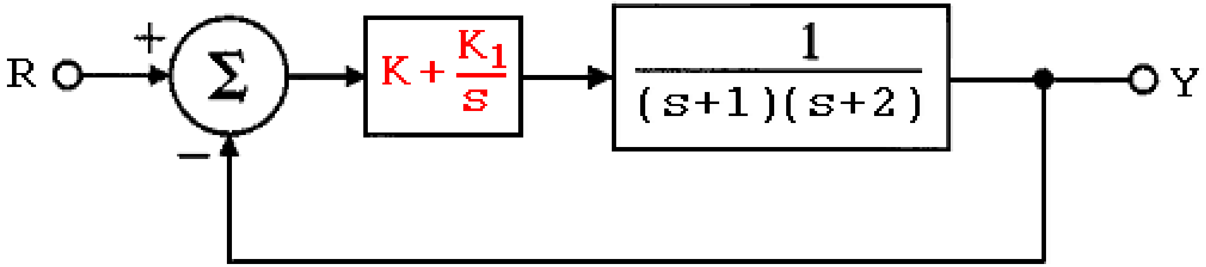
Example 6: Stability versus Two Parameter Range

Consider a Proportional-Integral (PI) control such as:



Find the range of the controller gains (K, K_1) so that the PI feedback system is stable.

Example 6 (cont'd)



- The characteristic equation for the system is given by:

$$1 + \left(K + \frac{K_1}{s} \right) \frac{1}{(s+1)(s+2)} = 0$$

Example 6 (cont'd)

$$1+\left(K+\frac{K_1}{s}\right)\frac{1}{(s+1)(s+2)}=0$$

- Expressing the characteristic equation in polynomial form, we obtain:

$$s^3+3s^2+(2+K)s+K_1=0$$

Example 6 (cont'd)

$$a(s) = s^3 + 5s^2 + (K-6)s + K$$

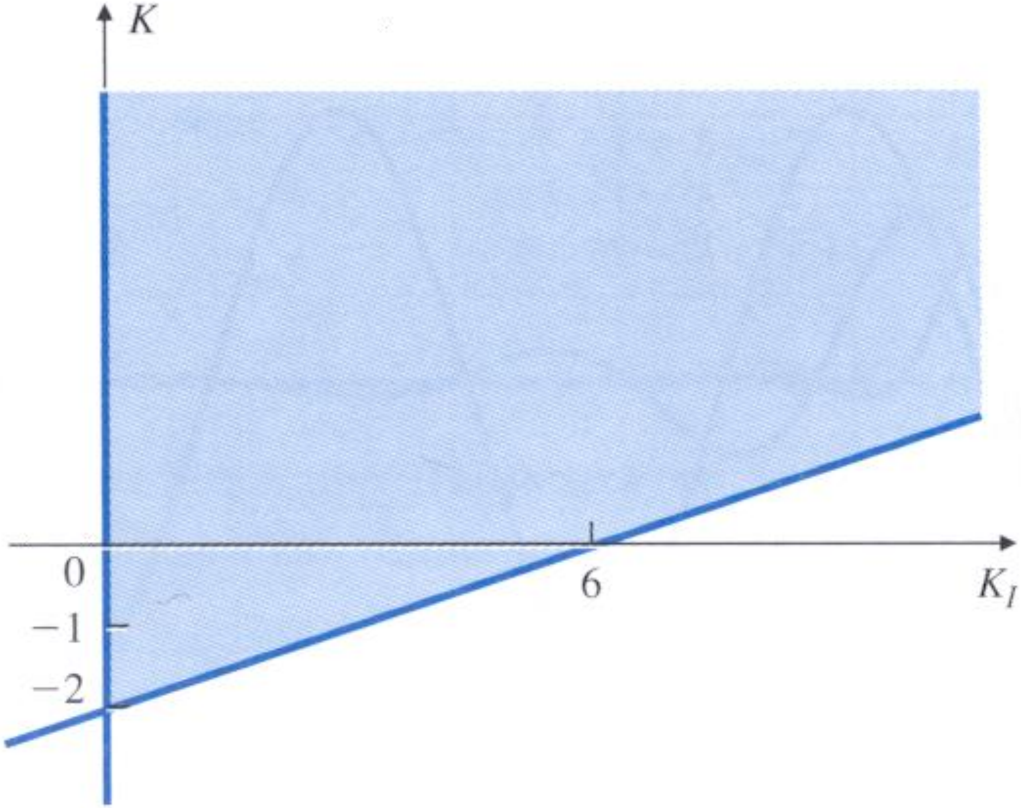
- The corresponding Routh array is:

$$\begin{array}{lll} s^3: & 1 & 2+K \\ s^2: & 3 & K_1 \\ s^1: & (6+3K-K_1)/3 & \\ s^0: & K_1 & \end{array}$$

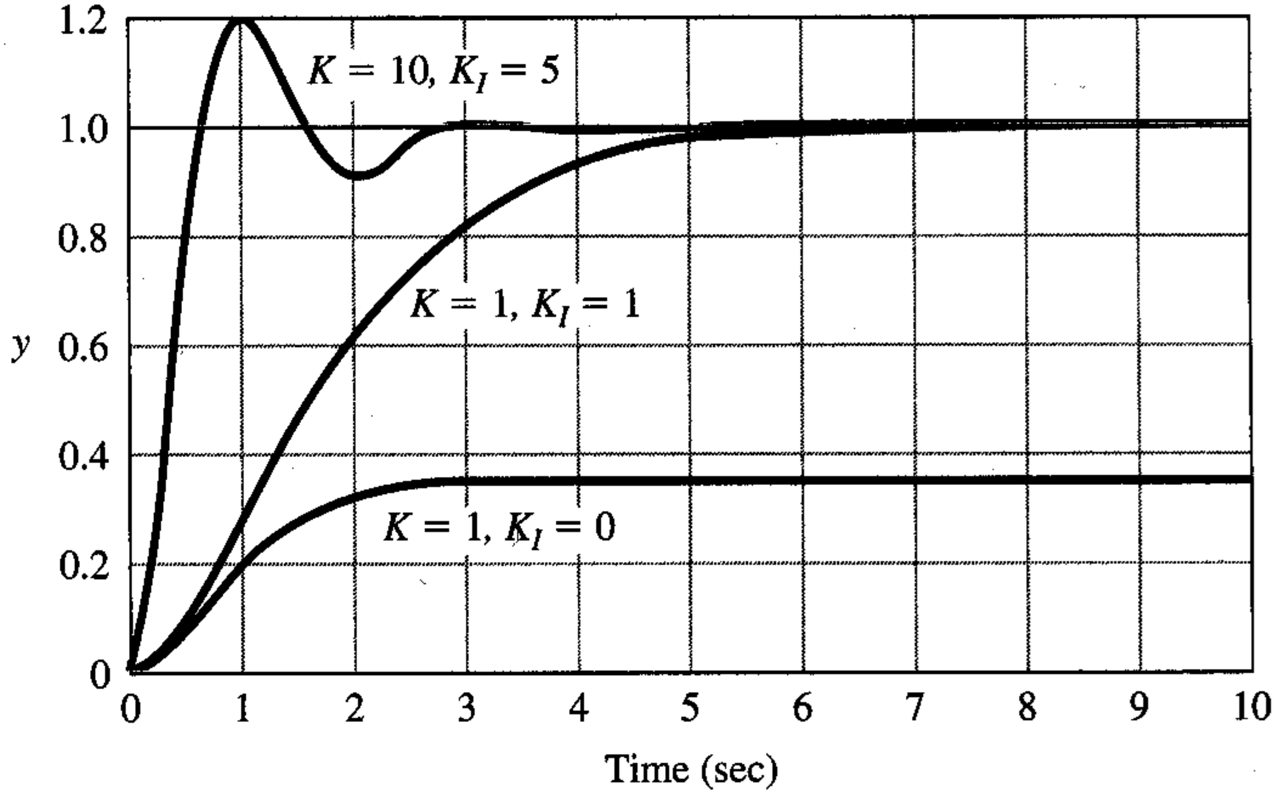
- For **stability**, we must have:

$$\boxed{K_1 > 0 \; and \; K > \frac{K_1}{3} - 2}$$

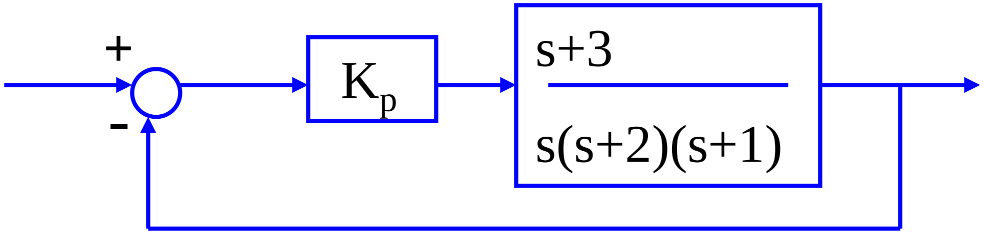
Example 6: Allowable region for Stability



Example 6: Transient Response for the System



Example 7:



Q : find range of K_p for stability

$$\text{Sol}: G_{c.l.}(s) = \frac{K_p(s+3)}{s(s+2)(s+1)+K_p(s+3)}$$

$$\therefore \text{char.poly } d(s) = s(s+2)(s+1)+K_p(s+3)$$

$$= s^3 + 3s^2 + (2+K_p)s + 3K_p$$

Example 7:

Routh table :

s^3 :	1	$2+K_p$
s^2 :	3	$3K_p$
s^1 :	$\dfrac{3(2+K_p)-3K_p}{3}$	
s^0 :	$3K_p$	

For A.S. : 1st col. > 0

$$\therefore \begin{cases} 3K_p > 0 \\ 3(2+K_p)-3K_p > 0 \end{cases}$$

$$\Rightarrow \boxed{K_p > 0}$$

Example 7:

e.g. $d(s) = s^3 + 3ks^2 + (k+2)s + 4$

remember 3rd order Routh criteria?

1) all coeff > 0

2) prod of mid two $>$ outer prod.

$\therefore$ For A.S. : we need :

1) $3k > 0 \qquad \Rightarrow k > 0$

$k + 2 > 0 \qquad \Rightarrow k > -2$

2) $3k(k+2) > 1 \times 4$

$$3k^2 + 6k - 4 > 0$$

$$k^2 + 2k - \frac{4}{3} > 0$$

$$(k+1)^2 - \frac{7}{3} > 0$$

Example 7:

$$(k+1)^2 > \frac{7}{3}$$

$$k+1 > \sqrt{\frac{7}{3}} \Rightarrow k > \sqrt{\frac{7}{3}} - 1$$

or $k+1 < -\sqrt{\frac{7}{3}} \Rightarrow k < -\sqrt{\frac{7}{3}} - 1$

but $k > 0$ and $k > -2$ also.

over all we need

$k \cancel{>} 0.5$

$$k > \sqrt{\frac{7}{3}} - 1 \approx 0.528$$

for A.S.

Example 7:

If we set $3k(k+2)=4$

we get : $k=\sqrt{\frac{7}{3}}-1$

At this $k, d(s)$ is M.S.

And s^1 row $=0$

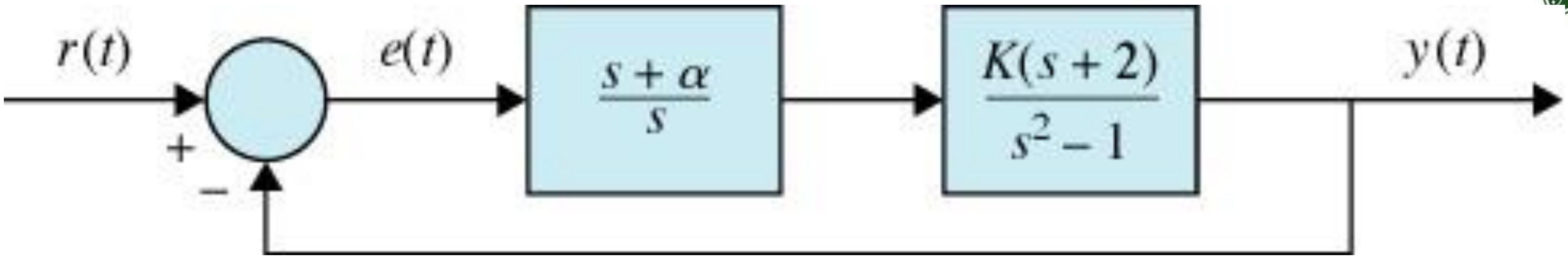
From s^2 row : $3ks^2+4=A(s)=0$

$$\Rightarrow s=\pm j\sqrt{\frac{4}{3k}}$$

leads to sustained oscillation

osci freq : $\omega=\sqrt{\frac{4}{3k}}=\sqrt{\frac{4}{3(\sqrt{\frac{7}{3}}-1)}}$

Example 8 : find region of stability in K-α plane



$$H(s)=\frac{(s+\alpha)K(s+2)}{s(s^2-1)+(s+\alpha)K(s+2)}$$

$$d(s)=s(s^2-1)+(s+\alpha)K(s+2)$$

$$=s^3+Ks^2+(K(\alpha+2)-1)s+2\alpha K$$

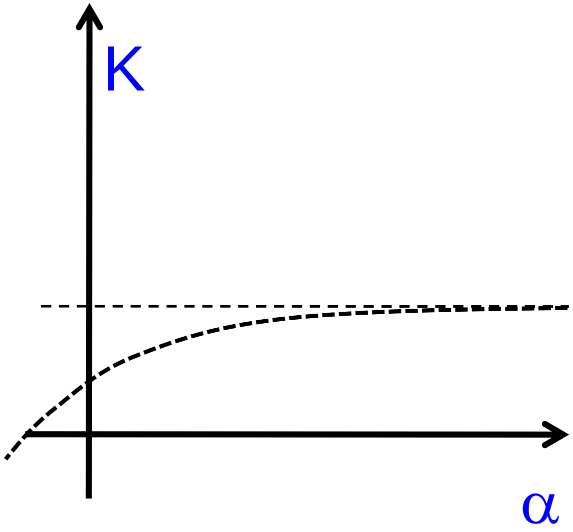
Routh Criteria :

$$K>0$$

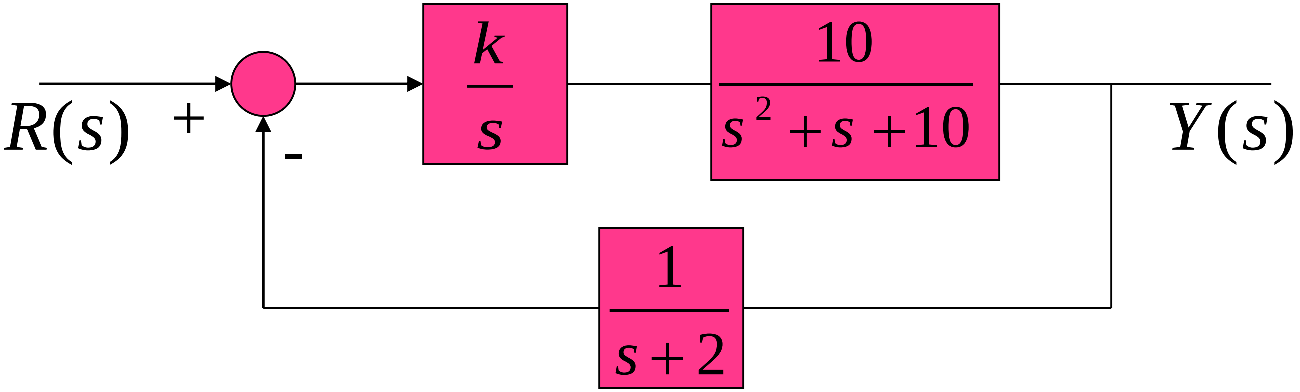
$$K(\alpha+2)-1>0 \qquad \Rightarrow K>\frac{1}{\alpha+2}$$

$$2\alpha K>0 \qquad \Rightarrow \alpha>0$$

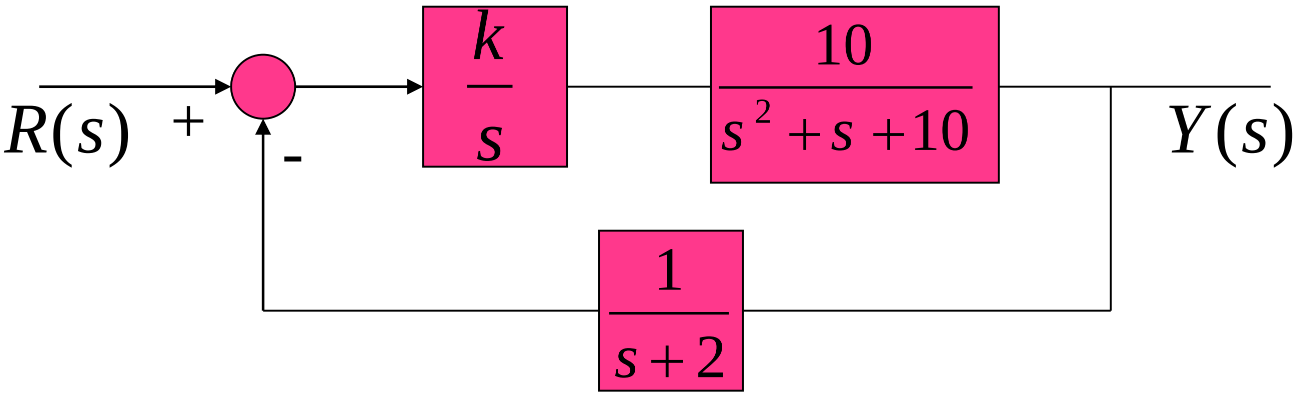
$$K(K(\alpha+2)-1)>2\alpha K \qquad \Rightarrow K>\frac{1+2\alpha}{\alpha+2}$$

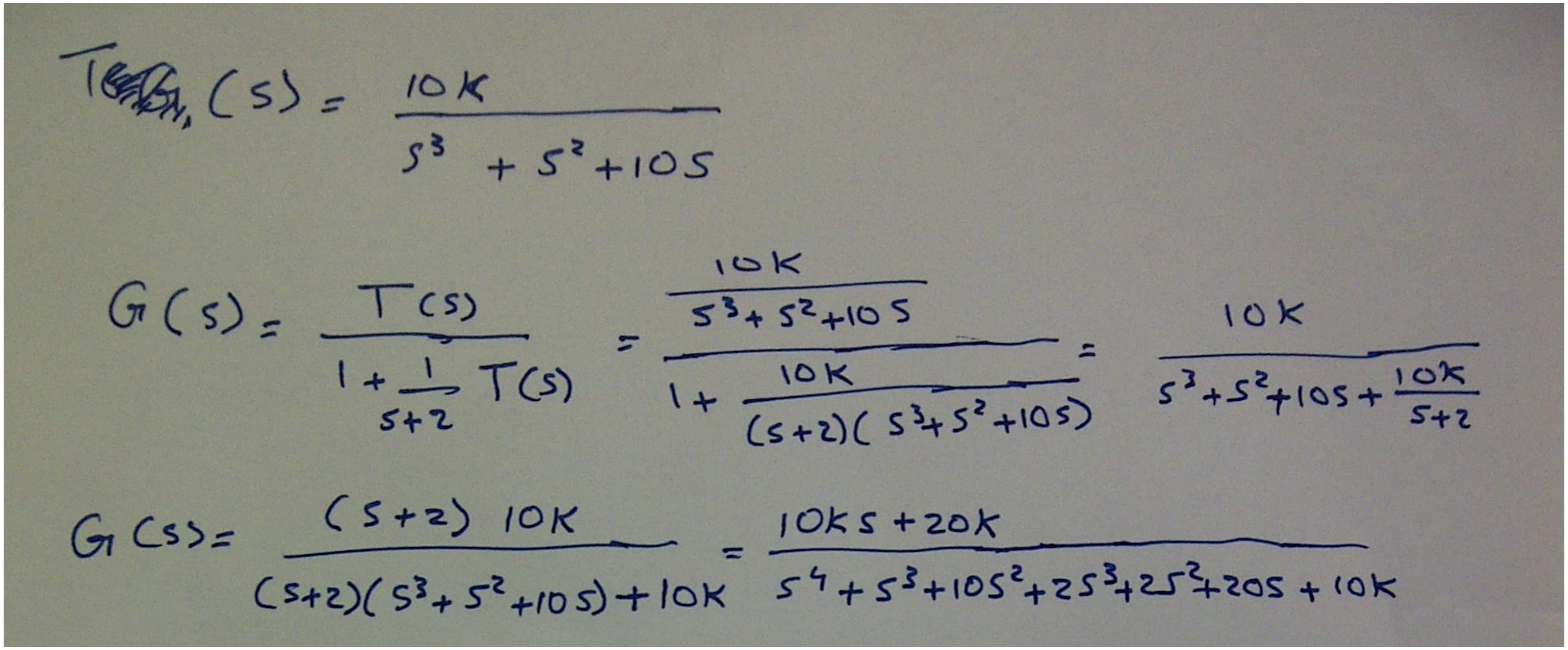


POP. Quiz:



Example:



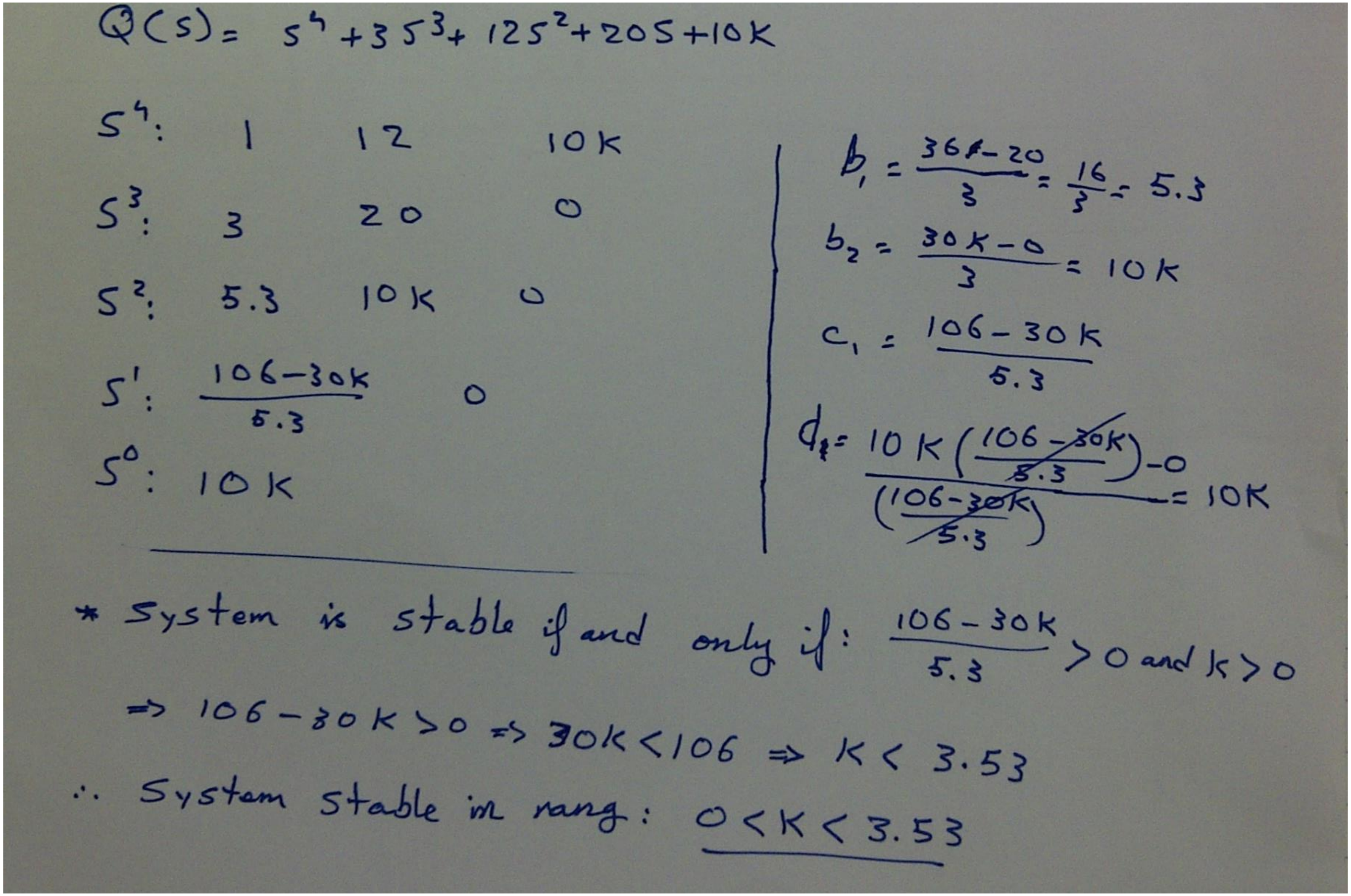


$$T(s) = \frac{10K}{s^3 + s^2 + 10s}$$

$$G(s) = \frac{T(s)}{1 + \frac{1}{s+2}T(s)} = \frac{\frac{10K}{s^3+s^2+10s}}{1 + \frac{10K}{(s+2)(s^3+s^2+10s)}} = \frac{10K}{s^3+s^2+10s+\frac{10K}{s+2}}$$

$$G(s) = \frac{(s+2)\,10K}{(s+2)(s^3+s^2+10s)+10K} = \frac{10Ks+20K}{s^4+s^3+10s^2+2s^3+2s^2+20s+10K}$$

Example:



$Q(s) = s^4 + 3s^3 + 12s^2 + 20s + 10K$

s^4:	1	12	10K
s^3:	3	20	0
s^2:	5.3	10K	0
s^1:	$\frac{106-30K}{5.3}$	0	
s^0:	10K		

$b_1 = \frac{36-20}{3} = \frac{16}{3} = 5.3$

$b_2 = \frac{30K-0}{3} = 10K$

$c_1 = \frac{106-30K}{5.3}$

$d_1 = \frac{10K\left(\cancel{\frac{106-30K}{5.3}}\right)-0}{\left(\cancel{\frac{106-30K}{5.3}}\right)} = 10K$

* System is stable if and only if: $\frac{106-30K}{5.3} > 0$ and $K > 0$

$\Rightarrow 106 - 30K > 0 \Rightarrow 30K < 106 \Rightarrow K < 3.53$

$\therefore$ System stable in rang: $\underline{0 < K < 3.53}$

END