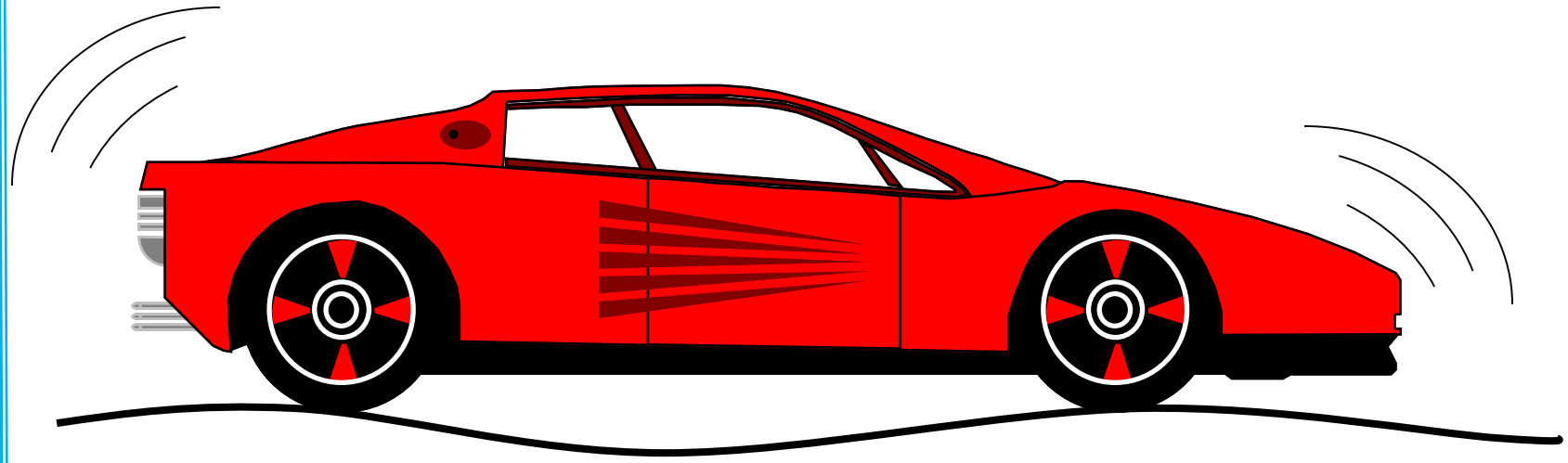
Multiple Degree of Freedom Systems

Chapters 5 and 6



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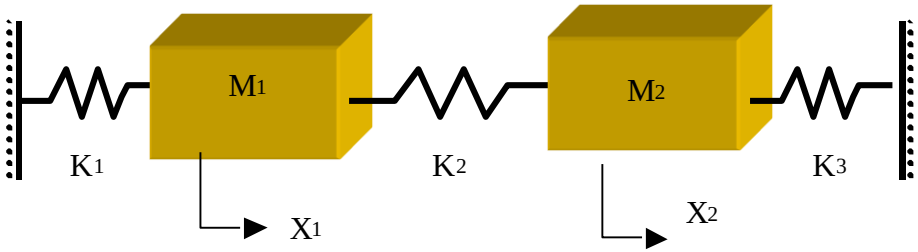
Multiple Degree of Freedom Systems

Chapters 5 and 6

A MDOF system is one that requires more than 1 independent coordinate to describe its motion.

The coordinates are called generalized coordinates when they are independent of each other.

The number of generalized coordinates are equal to the number of degrees of freedom of the system.



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Multiple Degree of Freedom Systems

A MDOF system that has ‘N’ degrees of freedom, has ‘N’ natural frequencies

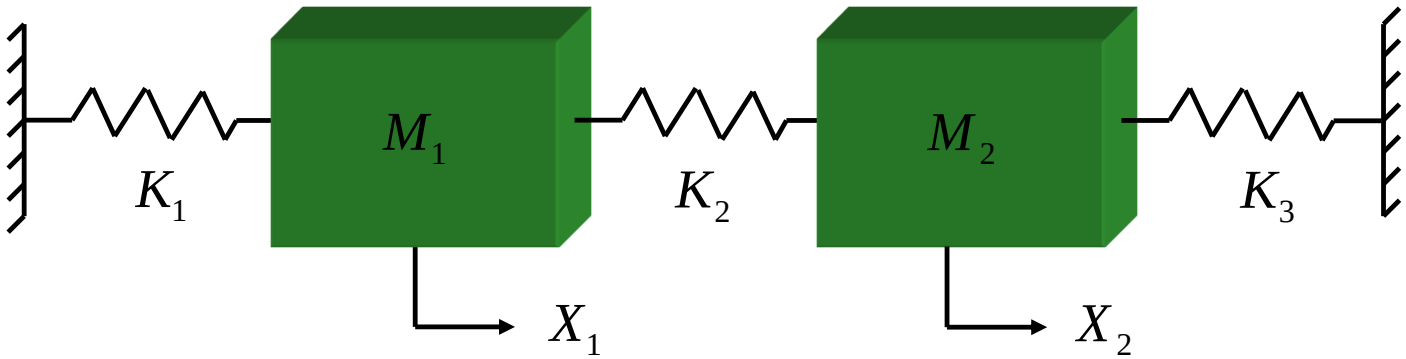
For each natural frequency there will be a natural state of vibration with a displacement configuration that is called a normal mode.

Mathematical terms corresponding to these quantities are called:

Eigenvalues (natural frequencies)

Eigenvectors (normal modes or mode shapes)

2 DOF Free Vibration



Given:

$$x_1(0) = 1 \qquad \dot{x}_1 = 0$$

$$x_2(0) = 2 \qquad \dot{x}_2 = 0$$

Determine:

$$x_1(t) = ?$$

$$x_2(t) = ?$$

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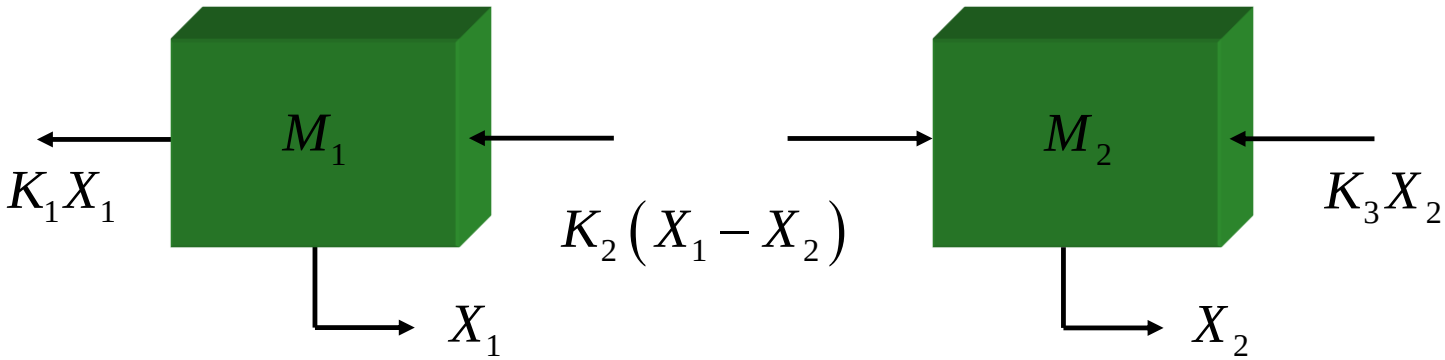
2 DOF Free Vibration: FBDs & EOM



Dynamic Equilibrium:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = 0$$

Matrix Form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

2 DOF Free Vibration: Solution

Assume a Solution:

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi)$$

Substitution:

$$-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi) + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Rearranging:

$$\left(-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

2 DOF Free Vibration: Solution

Let $\lambda = \omega^2$ and divide by $\sin(\omega t + \phi)$:

$$\begin{bmatrix} (k_1 + k_2) - m_1\lambda & -k_2 \\ -k_2 & (k_2 + k_3) - m_2\lambda \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For a non-trivial solution: $$\det \begin{bmatrix} (k_1 + k_2) - m_1\lambda & -k_2 \\ -k_2 & (k_2 + k_3) - m_2\lambda \end{bmatrix} = 0$$

Using numerical values: $k_1 = 10 \quad k_2 = 40 \quad k_3 = 5 \quad m_1 = 2 \quad m_2 = 5$

$$\det \begin{bmatrix} 50 - 2\lambda & -40 \\ -40 & 45 - 5\lambda \end{bmatrix} = 0$$

2 DOF Free Vibration: Eigen Solution

The solution becomes:

$$(50-2\lambda)(45-5\lambda)-1600=0 \longrightarrow 10\lambda^2-340\lambda+650=0$$

where: $\lambda_1 = 2.03$ **and** $\lambda_2 = 31.96$

Therefore: $\omega_1 = 1.43$ **and** $\omega_2 = 5.65 \quad \dfrac{\text{rad}}{\text{sec}}$

This means that there are two solutions and the total solution is the superposition of each solution.

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 \sin(\omega_1 t+\phi_1) + \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 \sin(\omega_2 t+\phi_2)$$

2 DOF Free Vibration: Mode Shapes

Using $\lambda_1 = 2.03$ $\longrightarrow$ $\begin{bmatrix} 50-2(2.03) & -40 \\ -40 & 45-2(2.03) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$\left(50-2(2.03)\right)X_1 - 40X_2 = 0$$

$$X_1 = 0.87X_2$$

$$\{X\}_1 = \begin{Bmatrix} 1 \\ 1.15 \end{Bmatrix}$$

mode shape for 1st natural frequency

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2 DOF Free Vibration: Mode Shapes

Using $\lambda_2 = 31.97$ $\longrightarrow$ $\begin{bmatrix} 50-2(31.97) & -40 \\ -40 & 45-2(31.97) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$\left(50-2\left(31.97\right)\right)X_1 - 40X_2 = 0$$

$$X_1 = -2.87X_2$$

$$\{X\}_2 = \begin{Bmatrix} 1 \\ -0.35 \end{Bmatrix}$$

mode shape for 2nd natural frequency

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2 DOF Free Vibration: Free Vibration Solution

Therefore the free vibration solution becomes:

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} A\{X\}_1 \sin(\omega_1 t + \phi_1) + B\{X\}_2 \sin(\omega_2 t + \phi_2)$$

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} A\begin{Bmatrix} 1 \\ 1.15 \end{Bmatrix}_1 \sin(\omega_1 t + \phi_1) + B\begin{Bmatrix} 1 \\ -0.35 \end{Bmatrix}_2 \sin(\omega_2 t + \phi_2)$$

$A,\ \phi_1,\ B,\ \phi_2$ **are determined from the initial conditions.**

2 DOF Free Vibration: Initial Conditions

$$x_1(t) = AX_{11}\sin(\omega_1 t + \phi_1) + BX_{12}\sin(\omega_2 t + \phi_2)$$

$$x_2(t) = AX_{21}\sin(\omega_1 t + \phi_1) + BX_{22}\sin(\omega_2 t + \phi_2)$$

$$\dot{x}_1(t) = A\omega_1 X_{11}\cos(\omega_1 t + \phi_1) + B\omega_2 X_{12}\cos(\omega_2 t + \phi_2)$$

$$\dot{x}_2(t) = A\omega_1 X_{21}\cos(\omega_1 t + \phi_1) + B\omega_2 X_{22}\cos(\omega_2 t + \phi_2)$$

Using

$$x_1(0) = 1 \qquad \dot{x}_1(0) = 0$$

$$x_2(0) = 2 \qquad \dot{x}_2(0) = 0$$

$$A = 1.566 \qquad B = -0.566$$

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2 DOF Free Vibration: Initial Conditions

$$\phi_1 = \frac{\pi}{2} \qquad \phi_2 = \frac{\pi}{2}$$

Therefore:

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = 1.566 \begin{Bmatrix} 1 \\ 1.15 \end{Bmatrix} \sin\left(1.43t + \frac{\pi}{2}\right) - 0.566 \begin{Bmatrix} 1 \\ -0.35 \end{Bmatrix} \sin\left(5.65t + \frac{\pi}{2}\right)$$

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} 1.566 \\ 1.801 \end{Bmatrix} \cos(1.43t) + \begin{Bmatrix} -0.566 \\ 0.1981 \end{Bmatrix} \cos(5.65t)$$

2 DOF Free Vibration: Initial Conditions

Eigen-solution method

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$\begin{bmatrix} 50 & -40 \\ -40 & 45 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \lambda \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

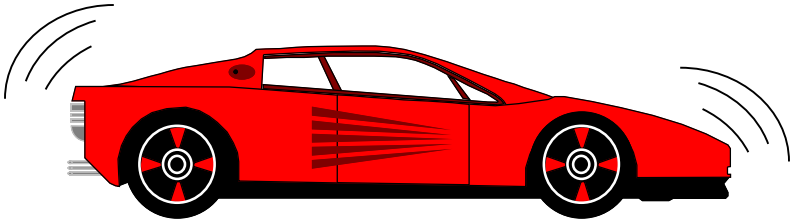
eigenvalues $\lambda_1 = 2.033 \quad \lambda_2 = 31.96$

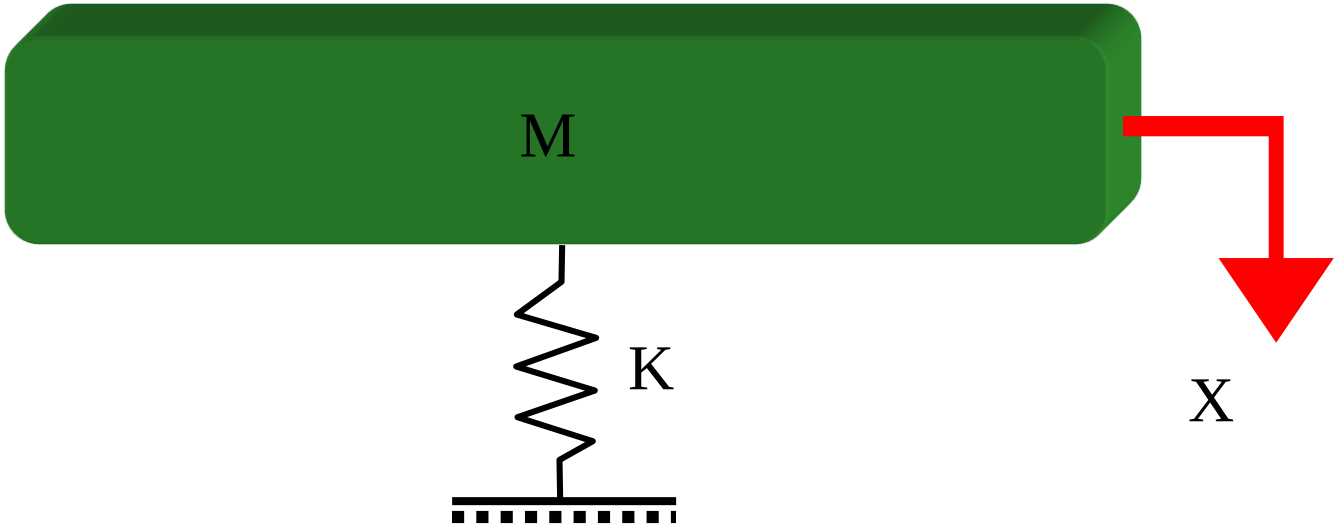
natural frequencies $\omega_1 = 1.426$ rad/sec $\omega_2 = 5.654$ rad/sec

Eigenvectors (mode shapes)

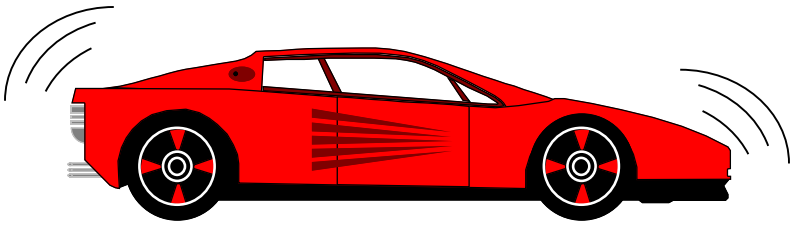
$$\begin{Bmatrix} X_1^1 \\ X_2^1 \end{Bmatrix} = \begin{Bmatrix} .3411 \\ .3917 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.15 \end{Bmatrix} \qquad \begin{Bmatrix} X_1^2 \\ X_2^2 \end{Bmatrix} = \begin{Bmatrix} .6194 \\ -.216 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -.348 \end{Bmatrix}$$

Vibrating Systems: Single Degree of Freedom (SDOF)

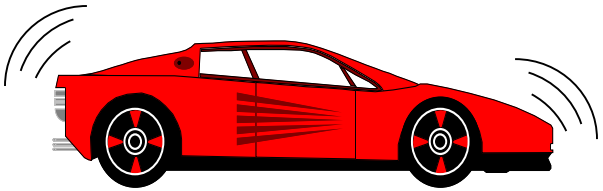


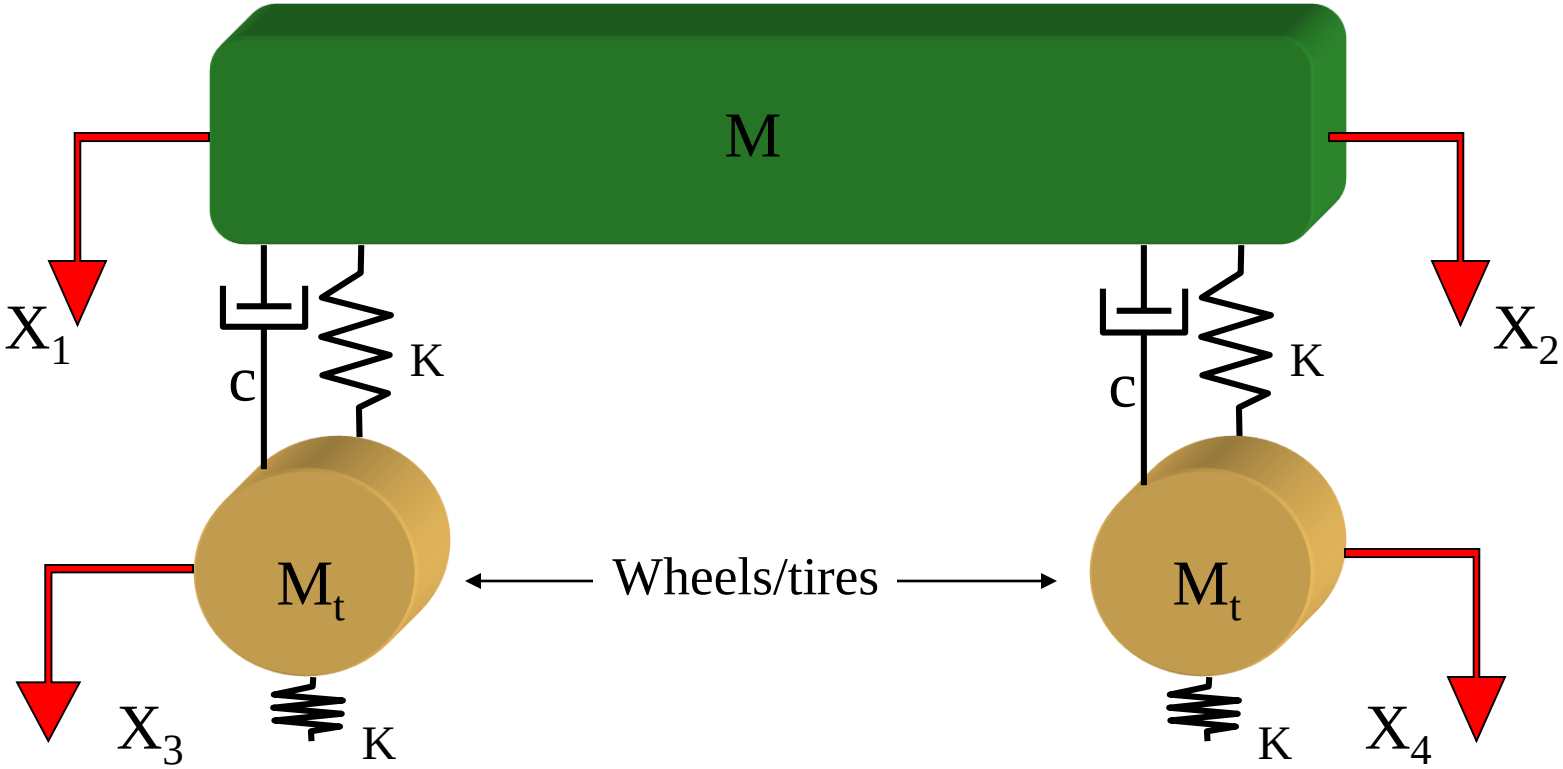


Vibrating Systems: Multi Degree of Freedom (MDOF)



Vibrating Systems: Multi Degree of Freedom (MDOF)





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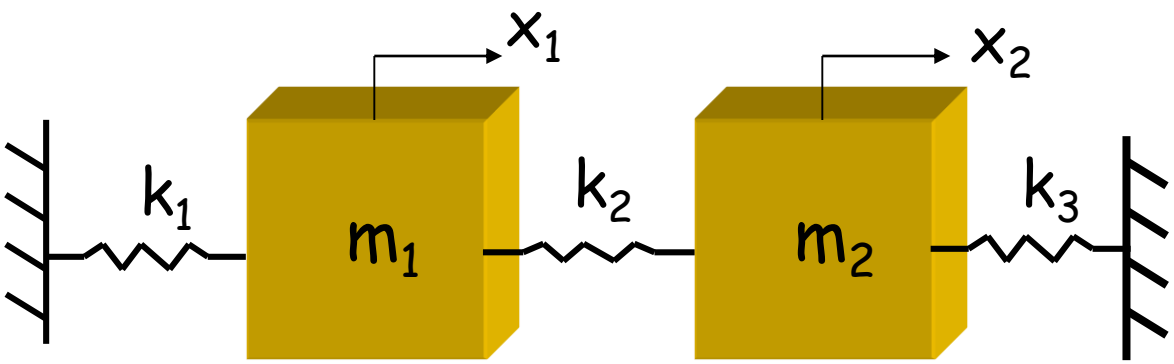
2 Degrees of Freedom: Free Vibration



Given a set of initial conditions, determine the free vibration response of the system.

$$x_1(0) \quad \dot{x}_1(0) \quad x_2(0) \quad \dot{x}_2(0)$$

$$x_1(t) = ? \qquad x_2(t) = ?$$

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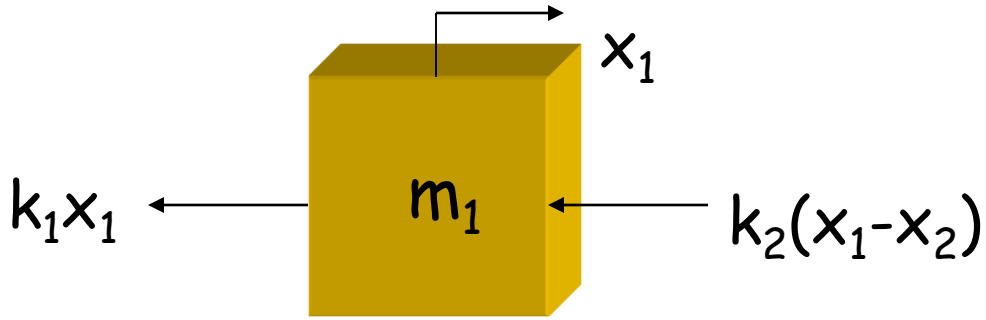
2 Degrees of Freedom: Free Vibration

F.B.D. for mass 1



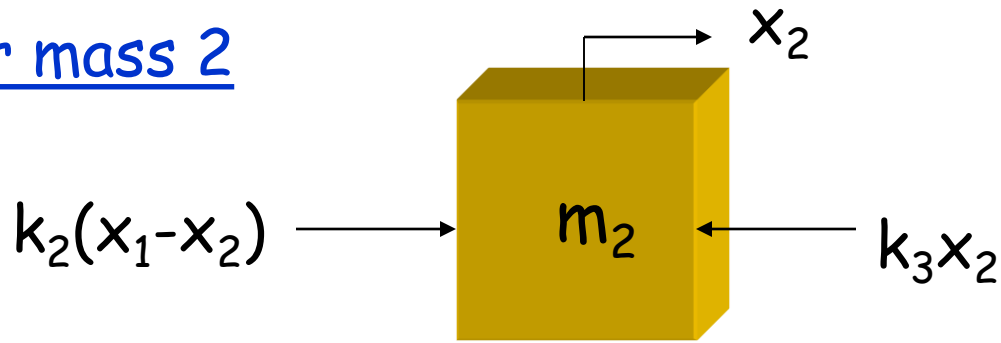
$$\sum F_{x_1} = \sum F_{eff} = m_1 \ddot{x}_1$$

$$-k_1 x - k_2 \left(x_1 - x_2 \right) = m_1 \ddot{x}_1$$

$$m_1 \ddot{x}_1 + \left(k_1 + k_2 \right) x_1 - k_2 x_2 = 0$$ EOM #1

2 Degrees of Freedom: Free Vibration

F.B.D. for mass 2



$$\sum F_{x_2} = \sum F_{eff}$$

$$k_2\left(x_1 - x_2\right) - k_3 x_2 = m_2 \ddot{x}_2$$

$$m_2 \ddot{x}_2 + \left(k_2 + k_3\right) x_2 - k_2 x_1 = 0$$ EOM #2

2 Degrees of Freedom: Free Vibration

$$m_1\ddot{x}_1 + \left(k_2 + k_1\right)x_1 - k_2 x_2 = 0$$

$$m_2\ddot{x}_2 + \left(k_2 + k_3\right)x_2 - k_2 x_1 = 0$$

2nd order differential equations

Linear

Homogeneous

Constant coefficients

Coupled

2 Degrees of Freedom: Free Vibration

For a solution to simultaneous equations, rewrite in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[m]\{\ddot{x}\} + [k]\{x\} = 0$$

Mass matrix

Acceleration vector

stiffness matrix

Displacement vector

2 Degrees of Freedom: Free Vibration

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

assume solution and substitute

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi)$$

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin(\omega t + \phi)$$

2 Degrees of Freedom: Free Vibration

$$-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}\sin(\omega t+\phi)+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}\sin(\omega t+\phi)=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\right)\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}\sin(\omega t+\phi)=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\right)\begin{Bmatrix} X_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2+k_3-\omega^2 m_2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

A solution to this set of equations exists if the determinant of the coefficient matrix is equal to zero.

2 Degrees of Freedom: Free Vibration

$$\det\begin{bmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2+k_3-\omega^2 m_2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(k_1+k_2-\omega^2 m_1\right)\left(k_2+k_3-\omega^2 m_2\right)-k_2^2=0$$

$$let: \lambda=\omega^2$$

$$\left(k_1+k_2-\lambda m_1\right)\left(k_2+k_3-\lambda m_2\right)-k_2^2=0$$

$$\left(k_1+k_2\right)\left(k_2+k_3\right)-\lambda m_2\left(k_1+k_2\right)-\lambda^2 m_1 m_2-k_2^2=0$$

$$m_1 m_2 \lambda^2-\lambda\left[m_2\left(k_1+k_2\right)+m_1\left(k_2+k_3\right)\right]+\left(k_1+k_2\right)\left(k_2+k_3\right)-k_2^2=0$$

2 Degrees of Freedom: Free Vibration

$$m_1 m_2 \lambda^2 - \lambda\left[m_2\left(k_1+k_2\right)+m_1\left(k_2+k_3\right)\right]+\left(k_1+k_2\right)\left(k_2+k_3\right)-k_2^2=0$$

A quadratic equation in λ

$$\lambda_{1,2}=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\lambda_{1,2}=\frac{-\left[m_2\left(k_1+k_2\right)+m_1\left(k_2+k_3\right)\right]}{2m_1m_2}\pm\frac{\sqrt{\left[m_2\left(k_1+k_2\right)+m_1\left(k_2+k_3\right)\right]^2-4m_1m_2\left(\left(k_1+k_2\right)\left(k_2+k_3\right)-k_2^2\right)}}{2m_1m_2}$$

Recall → $\lambda=\omega^2$

Therefore → $\omega_1=\sqrt{\lambda_1}$ $\quad\omega_2=\sqrt{\lambda_2}$

ω_1 and ω_2 are the natural frequencies of the system

2 Degrees of Freedom: Free Vibration

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin\left(\omega t + \phi\right)$$

for each value of ω_1 and ω_2 there is a corresponding vector $\begin{Bmatrix} X_1^1 \\ X_2^1 \end{Bmatrix}$ $\begin{Bmatrix} X_1^2 \\ X_2^2 \end{Bmatrix}$

$\begin{Bmatrix} X_1^1 \\ X_2^1 \end{Bmatrix}$ $\begin{Bmatrix} X_1^2 \\ X_2^2 \end{Bmatrix}$ are called the mode shapes of the system

The total solution is:

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = C_1 \begin{Bmatrix} X_1^1 \\ X_2^1 \end{Bmatrix} \sin\left(\omega_1 t + \phi_1\right) + C_2 \begin{Bmatrix} X_1^2 \\ X_2^2 \end{Bmatrix} \sin\left(\omega_2 t + \phi_2\right)$$

2 Degrees of Freedom: Free Vibration

Alternative Solution method: Eigen-Solution

$$\left(-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\right)\begin{Bmatrix} X_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} X_1 \\ x_2 \end{Bmatrix}=\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} X_1 \\ x_2 \end{Bmatrix}$$

Let $\lambda=\omega^2$

$$[k]\{x\}=\lambda[m]\{x\}$$

Eigenvalue problem

2 Degrees of Freedom: Free Vibration

$$[k]\{x\} = \lambda[m]\{x\}$$

eigenvalues = $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\lambda_N$ where N = the number of degrees of freedom

and associated eigenvectors $\begin{Bmatrix} X_1^1 \\ X_2^1 \end{Bmatrix}, \begin{Bmatrix} X_1^2 \\ X_2^2 \end{Bmatrix}, \begin{Bmatrix} X_1^3 \\ X_2^3 \end{Bmatrix}, \ldots\ldots\ldots\ldots \begin{Bmatrix} X_1^N \\ X_2^N \end{Bmatrix}$

The natural frequencies are the square root of the eigenvalues

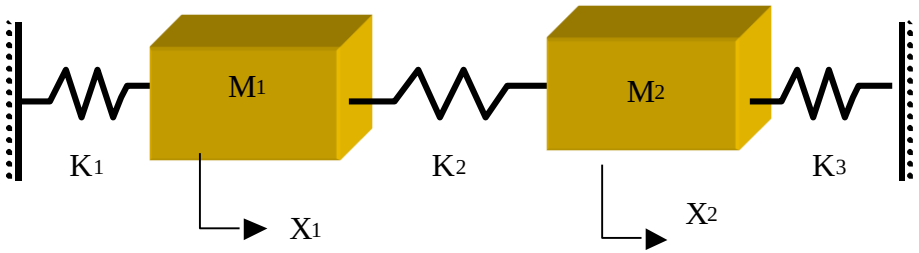
$$\omega_i = \sqrt{\lambda_i}$$

The eigenvectors are also called the mode shapes of the system

Example: 2 DOF System

Free Vibration

No damping



$k_1 = 16$ N/m $k_2 = 24$ N/m $k_3 = 16$ N/m $m_1 = 1$ kg $m_2 = 1$ kg

Given:

$$x_1(0) = 1 \qquad \dot{x}_1(0) = 0$$
$$x_2(0) = 2 \qquad \dot{x}_2(0) = 0$$

Determine:

$$x_1(t) = ?$$
$$x_2(t) = ?$$

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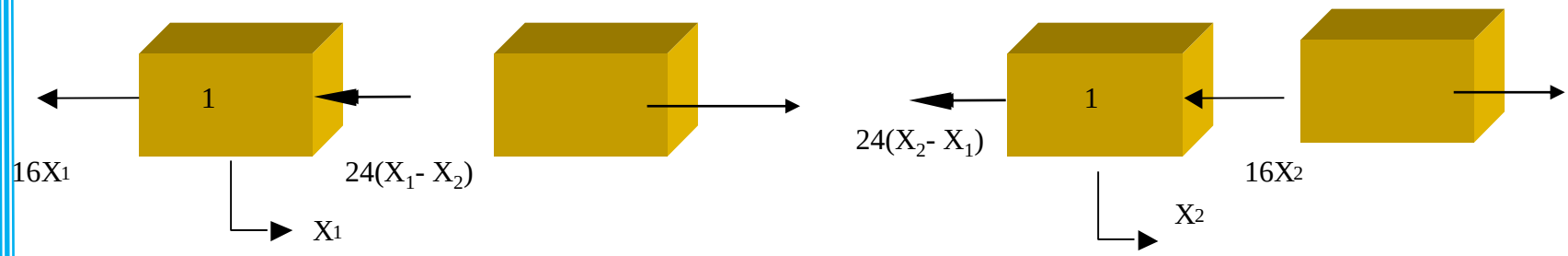
Draw a FBD for each mass and apply Newton' second law



For mass 1

$$-24\left(x_1 - x_2\right) - 16x_1 = \ddot{x}_1$$

$$\ddot{x}_1 + (40)x_1 - 24x_2 = 0$$

For mass 2

$$24\left(x_1 - x_2\right) - 16x_2 = \ddot{x}_2$$

$$\ddot{x}_2 + (40)x_2 - 24x_1 = 0$$

These two equations are coupled and can be put into matrix form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 40 & -24 \\ -24 & 40 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\ddot{x}_1 + (40)x_1 - 24x_2 = 0 \qquad \ddot{x}_2 + (40)x_2 - 24x_1 = 0$$

To solve these equations assume a solution of the form

$$x_1(t) = C_1 \sin(\omega t + \phi)$$
$$x_2(t) = C_2 \sin(\omega t + \phi)$$

Substituting into the equations of motion

$$\left(-\omega^2 C_1 + 40C_1 - 24C_2\right)\sin(\omega t + \phi) = 0$$
$$\left(-\omega^2 C_2 - 24C_1 + 40C_2\right)\sin(\omega t + \phi) = 0$$

@drsaeedasiri

$$\left(40-\omega^2\right)C_1-24C_2=0$$

$$-24C_1+\left(40-\omega^2\right)C_2=0$$

The same solution is obtained from the matrix form also

$$\begin{bmatrix} 40-\omega^2 & -24 \\ -24 & 40-\omega^2 \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

These equations are identical

Using this form of the equations

$$\left(40-\omega^2\right)C_1-24C_2=0$$

$$-24C_1+\left(40-\omega^2\right)C_2=0$$

$$\frac{C_2}{C_1}=\frac{24}{40-\omega^2} \quad \text{or} \quad \frac{C_2}{C_1}=\frac{40-\omega^2}{24}$$

therefore

$$\frac{24}{40-\omega^2}=\frac{40-\omega^2}{24}$$

or $\left(40-\omega^2\right)^2-24^2=0$ which can be expressed as $\left(40-\omega^2+24\right)\left(40-\omega^2-24\right)=0$

$\left(64-\omega^2\right)\left(16-\omega^2\right)=0$ which means $\omega^2=16$ and $\omega^2=64$

$\omega=\pm 4$ rad/sec or $\omega=\pm 8$ rad/sec

Therefore $\omega = 4$ rad/sec and $\omega = 8$ rad/sec

Let $\omega_1 = 4$ rad/sec and $\omega_2 = 8$ rad/sec

These are the two natural frequencies of the system

To determine C_1 and C_2 (the mode shapes) consider either equation

$$\left(40-\omega^2\right)C_1 - 24C_2 = 0$$

$$-24C_1 + \left(40-\omega^2\right)C_2 = 0$$

$$\left(40-\omega^2\right)C_1 - 24C_2 = 0$$

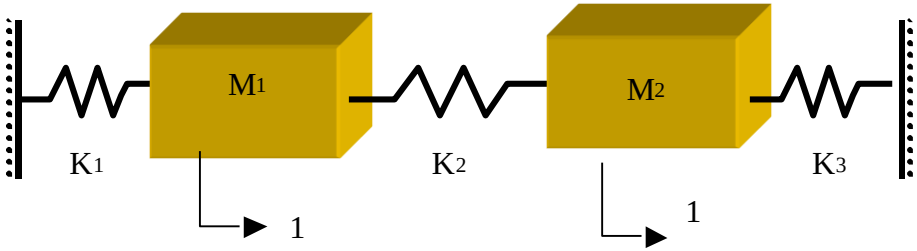
For $\omega^2 = \omega_1^2 = 16$, $\dfrac{C_2}{C_1} = 1$ or $C_2 = C_1$

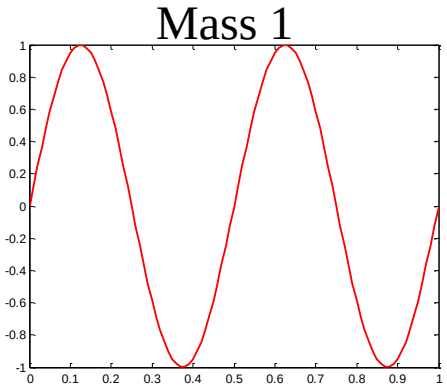
For $\omega^2 = \omega_2^{\ 2} = 64$, $\dfrac{C_2}{C_1} = -1$ or $C_2 = -C_1$

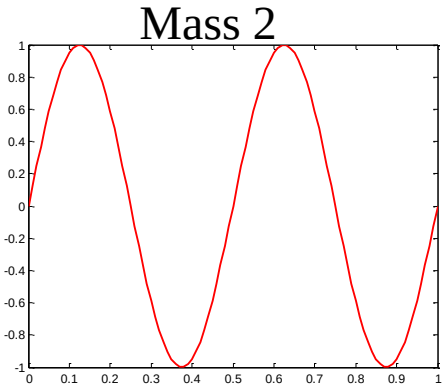
Physical Interpretation

For $\omega = \omega_1 = 4$ rad/sec $\quad C_2 = C_1$

Hence the two masses have the same amplitude and are in phase



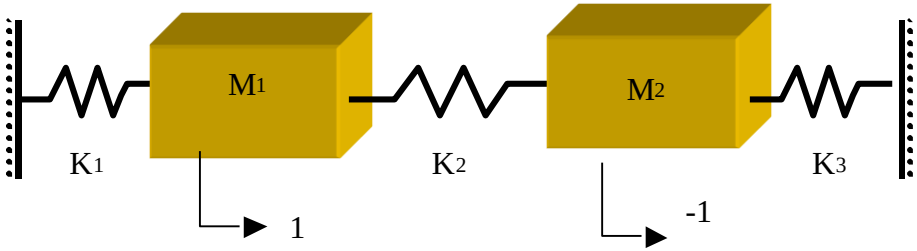


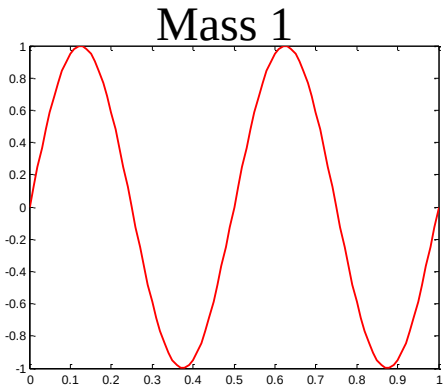


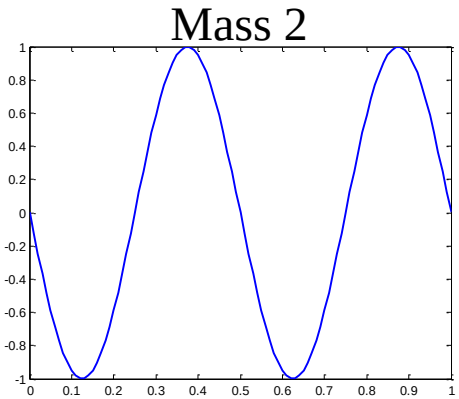
Physical Interpretation

For $\omega = \omega_1 = 8$ rad/sec $\quad C_2 = -C_1$

Hence the two masses have the same amplitude and are out of phase







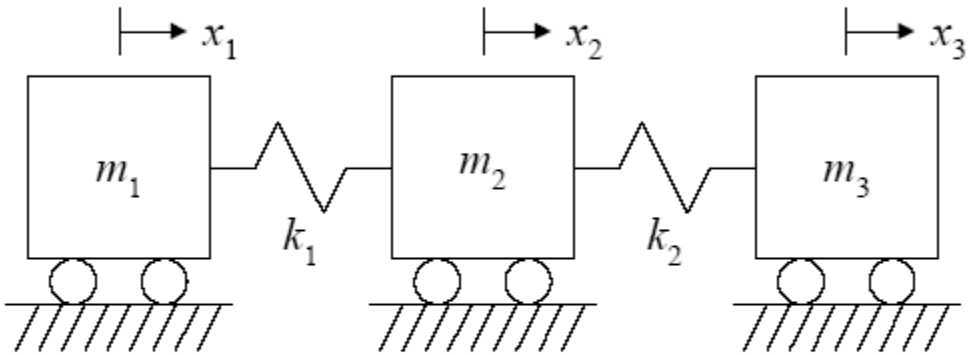
Rigid Body Motion

- Positive Definite Systems
  - Inertia matrix is positive definite
  - Stiffness matrix is positive definite
  - Harmonic oscillations in all modes

- Positive Semidefinite Systems
  - Inertia matrix is positive definite
  - Stiffness matrix is positive semidefinite
  - Rigid body motion

Rigid Body Motion

- Typical System with Rigid Body Motion



  - Kinetic energy

$$T = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 + \tfrac{1}{2} m_3 \dot{x}_3^2 = \tfrac{1}{2} \{\dot{x}\}^T [m] \{\dot{x}\}$$

  - Potential energy

$$V = \tfrac{1}{2} k_1 (x_2 - x_1)^2 + \tfrac{1}{2} k_2 (x_3 - x_2)^2 = \tfrac{1}{2} \{x\}^T [k] \{x\}$$

Rigid Body Motion

- Typical System with Rigid Body Motion (cont'd)
  - Equations of motion

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$$

  - Assumed solution

$$\{x(t)\} = \{X\} f(t)$$

Rigid Body Motion

- Typical System with Rigid Body Motion (cont'd)
  - Eigenvalue problem

$$[k]\{X\}-\omega^2[m]\{X\}=0$$

- Characteristic equation

$$\begin{vmatrix} k_1-\omega^2 m_1 & -k_1 & 0 \\ -k_1 & k_1+k_2-\omega^2 m_2 & -k_2 \\ 0 & -k_2 & k_2-\omega^2 m_3 \end{vmatrix}=0$$

$$\left(k_1-\omega^2 m_1\right)\left(k_1+k_2-\omega^2 m_2\right)\left(k_2-\omega^2 m_3\right)$$
$$-k_1^2\left(k_2-\omega^2 m_3\right)-k_2^2\left(k_1-\omega^2 m_1\right)=0$$

Rigid Body Motion

- Typical System with Rigid Body Motion (cont'd)
  - Zero eigenvalue
    - Rigid body mode
    - Eigenvector

$$\{X\} = \{X\}_0 = X_0\{1\}$$

$$\{X\}_0^T [k] \{X\}_0 = 0$$

$$\{X\}_0^T [m] \{X\}_0 > 0$$

    - Semidefinite systems have at least one zero eigenvalue

Rigid Body Motion

- Orthogonality of Rigid Body Modes
  - Rigid body mode must be orthogonal to all other modes

$$\{X\}_0^T[m]\{X\} = X_0(m_1X_1 + m_2X_2 + m_3X_3) = 0$$

$$m_1X_1 + m_2X_2 + m_3X_3 = 0$$

  - Momentum is zero

$$x_k(t) = X_k f(t); \quad k = 1,2,3$$

$$\dot{x}_k(t) = X_k \dot{f}(t); \quad k = 1,2,3$$

$$m_1\dot{x}_1(t) + m_2\dot{x}_2(t) + m_3\dot{x}_3(t) = 0$$

Rigid Body Motion

- Unrestrained Systems
  - Characteristics
    - Rigid body motion
    - Elastic motion
    - Stiffness matrix is singular
    - Flexibility matrix is undefined
    - Can be transformed to a fully restrained system
  - Transformation of coordinates
    - Constraint equation

$$m_1 x_1 + m_2 x_2 + m_3 x_3 = 0$$

$$x_3 = -\frac{m_1}{m_3} x_1 - \frac{m_2}{m_3} x_2$$

Rigid Body Motion

- Unrestrained Systems (cont'd)
  - Transformation of coordinates (cont'd)
    - Constraint equation (cont'd)

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m_1/m_3 & -m_2/m_3 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -m_1/m_3 & -m_2/m_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [c] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

    - Transformed kinetic energy

$$T = \tfrac{1}{2}\{\dot{x}\}_c^T [m] \{\dot{x}\}_c = \tfrac{1}{2}\{\dot{x}\}^T [c]^T [m][c] \{\dot{x}\} = \tfrac{1}{2}\{\dot{x}\}^T [m'] \{\dot{x}\}$$

Rigid Body Motion

- Unrestrained Systems (cont'd)
  - Transformation of coordinates (cont'd)
    - Transformed potential energy

$$V = \tfrac{1}{2}\{x\}_c^T[k]\{x\}_c = \tfrac{1}{2}\{x\}^T[c]^T[k][c]\{x\} = \tfrac{1}{2}\{x\}^T[k']\{x\}$$

    - Transformed mass matrix

$$[m'] = \frac{1}{m_3}\begin{bmatrix} m_1(m_1+m_3) & m_1 m_2 \\ m_1 m_2 & m_2(m_2+m_3) \end{bmatrix}$$

    - Transformed stiffness matrix

$$[k'] = \frac{1}{m_3^2}\begin{bmatrix} k_1 m_3^2 + k_2 m_1^2 & -k_1 m_3^2 + k_2 m_1(m_2+m_3) \\ -k_1 m_3^2 + k_2 m_1(m_2+m_3) & (k_1+k_2)m_3^2 + k_2 m_2(m_2+2m_3) \end{bmatrix}$$

Rigid Body Motion

- Unrestrained Systems (cont'd)
  - Transformation of coordinates (cont'd)
    - Eigenvalue problem

$$\omega^2[m']\{X\}=[k']\{X\}$$

    - Elastic modes

$$\{X\}_{1c}=[c]\{X\}_1$$

$$\{X\}_{2c}=[c]\{X\}_2$$

    - Rigid body mode

$$\{X\}_0=\{1\}$$

Rigid Body Motion

- Example of Rigid Body Motion

$$m_1 = m_2 = m_3 = m\,; \quad k_1 = k_2 = k$$

  - Equations of motion

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix}_c + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_c = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

  - Constraint transformation

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Rigid Body Motion

- Example of Rigid Body Motion (cont'd)
  - Transformed equations of motion

$$m\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

  - Eigenvalues and eigenvectors

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \{X\}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\omega_2 = \sqrt{\frac{3k}{m}} \qquad \{X\}_2 = \begin{Bmatrix} 0.5 \\ -1 \end{Bmatrix}$$

Rigid Body Motion

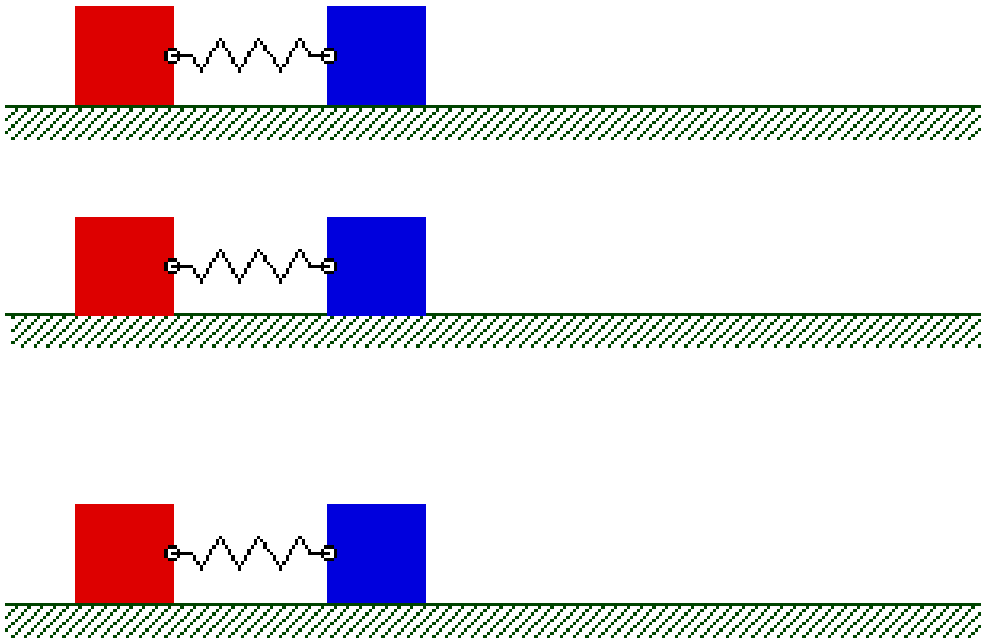
- Example of Rigid Body Motion (cont'd)
  - Rigid body motion

$$\omega_0 = 0 \qquad \{X\}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

  - Constrained eigenvectors

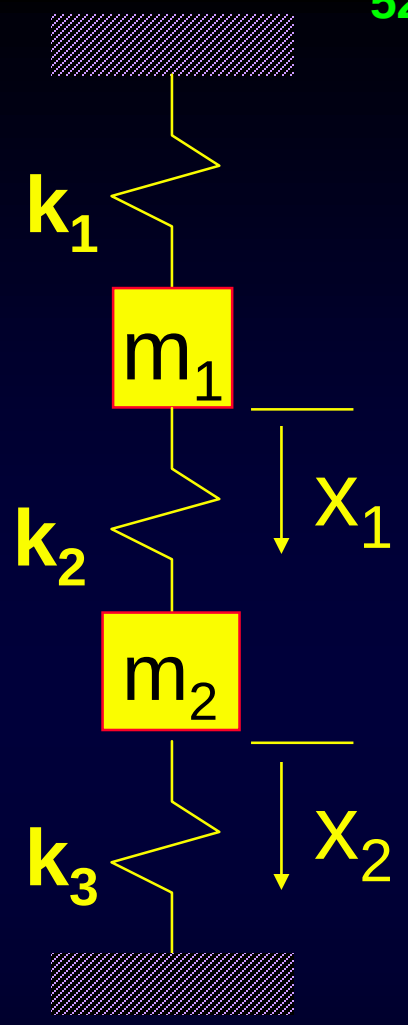
$$\{X\}_{1c} = [c]\{X\}_1 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \qquad \{X\}_{2c} = [c]\{X\}_2 = \begin{Bmatrix} 0.5 \\ -1 \\ 0.5 \end{Bmatrix}$$

Rigid body mode



First mode (rigid body mode - no strain in spring)
Second mode (vibrational mode - particles 180 degrees out of phase)
Total response (combination of rigid body mode and vibational mode)

2 DOF System



$$\sum F = m\ddot{x}, \downarrow +$$

Write equations for each DOF (mass):

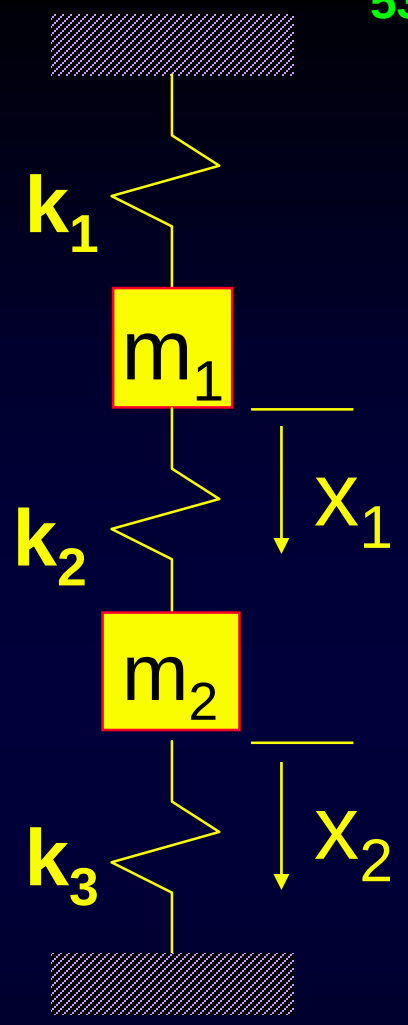
$$-k_1 x_1 + k_2 (x_2 - x_1) = m_1 \ddot{x}_1$$

$$-k_2 (x_2 - x_1) - k_3 x_2 = m_2 \ddot{x}_2$$

Or, in matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

2 DOF System



With 2 DOF, we intuitively expect two natural frequencies for the system.

From our previous work, we know that one solution for $x_1(t)$ is:

$$x_1(t) = X_{1m}\sin(\omega_n t + \phi), \text{ so}$$

$$\ddot{x}_1(t) = -\omega_n^2 X_{1m}\sin(\omega_n t + \phi)$$

So the previous matrices can be combined, as shown on the next slide.

2 DOF System - Matrix Representation

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

mass matrix | accel. vector | stiffness matrix | position vector | force vector

- In general, the response of each mass is a combination of motion at both natural frequencies.
- A *principal mode* exists in which both coordinates have harmonic motion of the same frequency, ω_p and the phase angle, ϕ is zero. We use this concept to find the principal frequencies, the ω_p's.

$$x_1(t) = X_{1m} \sin \omega_p t, \;\; \ddot{x}_1(t) = -X_{1m} \omega_p^2 \sin \omega_p t$$

$$x_2(t) = X_{2m} \sin \omega_p t, \;\; \ddot{x}_2(t) = -X_{2m} \omega_p^2 \sin \omega_p t$$

Principal Modes

- Again, a principal mode occurs when each mass oscillates with the same natural frequency.
- In a 2 DOF system, the masses will either move in together (in-phase) or directly apart (out-of-phase)
- In 3+ DOF systems, there can be different combinations of motion.
- The natural frequencies (ω_p's) are the frequencies at which principal mode oscillation occurs.
- At frequencies other than the natural frequencies, the system motion will be a combination of motion of some or all of the natural frequencies of the system

Simplification of Matrix

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} -\omega_p^2 X_{1m} \sin \omega_p t \\ -\omega_p^2 X_{2m} \sin \omega_p t \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_{1m} \sin \omega_p t \\ X_{2m} \sin \omega_p t \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Now, divide out $\sin\omega_p t$:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} -\omega_p^2 X_{1m} \\ -\omega_p^2 X_{2m} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_{1m} \\ X_{2m} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Combine Matrices

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} -\omega_p^2 X_{1m} \\ -\omega_p^2 X_{2m} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_{1m} \\ X_{2m} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

This can be combined into a single matrix:

$$\begin{bmatrix} k_1 + k_2 - m_1\omega_p^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2\omega_p^2 \end{bmatrix} \begin{Bmatrix} X_{1m} \\ X_{2m} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

We can solve this matrix by setting its determinant to zero:

Take Determinant, Solve for ω_1 & ω_2

$$\det\begin{bmatrix} k_1 + k_2 - m_1\omega_p^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2\omega_p^2 \end{bmatrix} = 0$$

$$\left(k_1 + k_2 - m_1\omega_p^2\right)\left(k_2 + k_3 - m_2\omega_p^2\right) - k_2^2 = 0$$

which expands to:

$$m_1 m_2 \omega_p^4 - \omega_p^2\left[m_1\left(k_2 + k_3\right) + m_2\left(k_1 + k_2\right)\right]$$

$$+ k_1 k_2 + k_2 k_3 + k_1 k_3 + k_2^2 - k_2^2 = 0$$

Solve for 2 values of ω_p^2 using the quadratic equation

Defining Mode Fractions

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} -\omega_p^2 X_{1m} \\ -\omega_p^2 X_{2m} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_{1m} \\ X_{2m} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(k_1 + k_2 - m_1\omega_p^2\right)X_{1m} - k_2 X_{2m} = 0$$

$$-k_2 X_{1m} + \left(k_2 + k_3 - m_2\omega_p^2\right)X_{2m} = 0$$

$$\text{Mode Fraction, } \mathrm{X_F} = \frac{X_{2m}}{X_{1m}}$$

$$\mathrm{X_F} = \frac{X_{2m}}{X_{1m}} = \frac{k_1 + k_2 - m_1\omega_p^2}{k_2} = \frac{k_2}{k_2 + k_3 - m_2\omega_p^2}$$

Plug in ω_{p1} & ω_{p2} Solve for 2 Mode Fractions

Our solution for ω_p^2 gave us two values. When we plug these two frequencies in for the mode fractions, we get two mode fractions or mode shapes:

$$X_F^{(1)} = \frac{X_{2m}^{(1)}}{X_{1m}^{(1)}} = \frac{k_1 + k_2 - m_1\omega_{p1}^2}{k_2} = \frac{k_2}{k_2 + k_3 - m_2\omega_{p1}^2}$$

$$X_F^{(2)} = \frac{X_{2m}^{(2)}}{X_{1m}^{(2)}} = \frac{k_1 + k_2 - m_1\omega_{p2}^2}{k_2} = \frac{k_2}{k_2 + k_3 - m_2\omega_{p2}^2}$$

Example

Let $m_1 = m_2 = m$, $k_1 = k_2 = k_3 = k$

$$m_1 m_2 \omega_p^4 - \omega_p^2 \left[m_1 (k_2 + k_3) + m_2 (k_1 + k_2) \right]$$

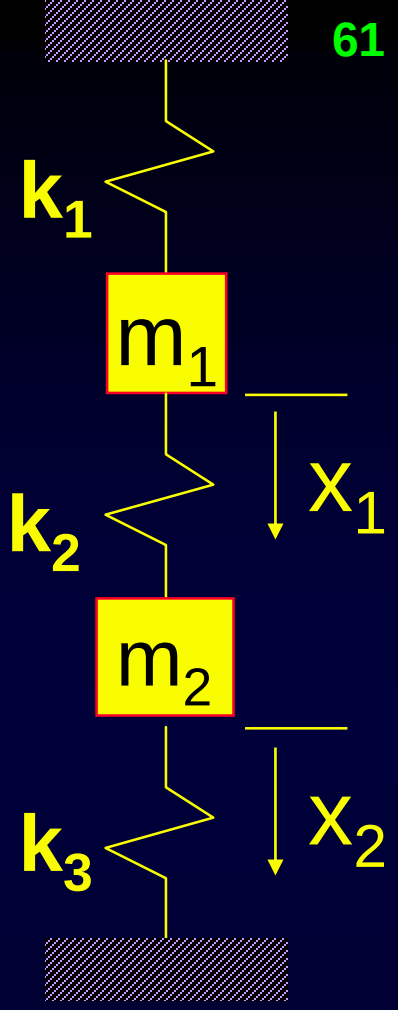
$$+ k_1 k_2 + k_2 k_3 + k_1 k_3 + k_2^2 - k_2^2 =$$

$$m^2 \omega_p^4 - \omega_p^2 \left[m(2k) + m(2k) \right]$$

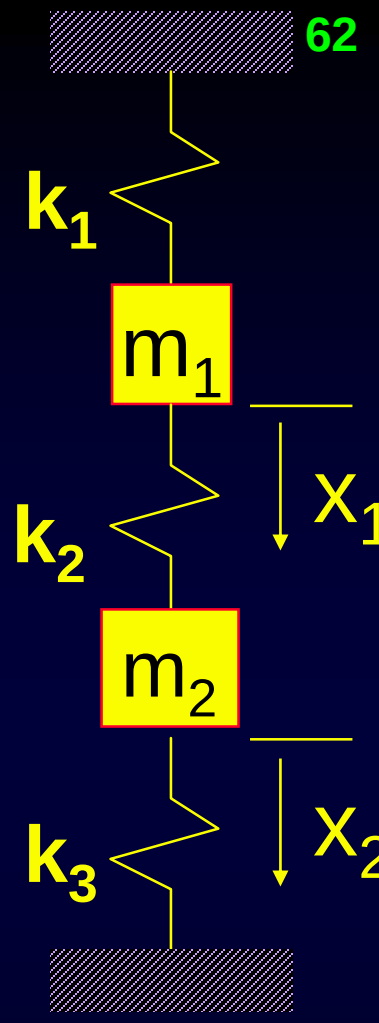
$$+ k^2 + k^2 + k^2 + k^2 - k^2 =$$

$$m^2 \omega_p^4 - 4km\omega_p^2 + 3k^2 = 0$$

$$\omega_p^4 - 4(k/m)\omega_p^2 + 3k^2 / m^2 = 0$$



Example: Natural Frequencies



$$\omega_p^4 - 4(k/m)\omega_p^2 + 3k^2/m^2 = 0$$

$$\omega_{p1,2}^2 = \frac{1}{2}\left[4k/m \pm \sqrt{(-4k/m)^2 - 4\left(3k^2/m^2\right)}\right]$$

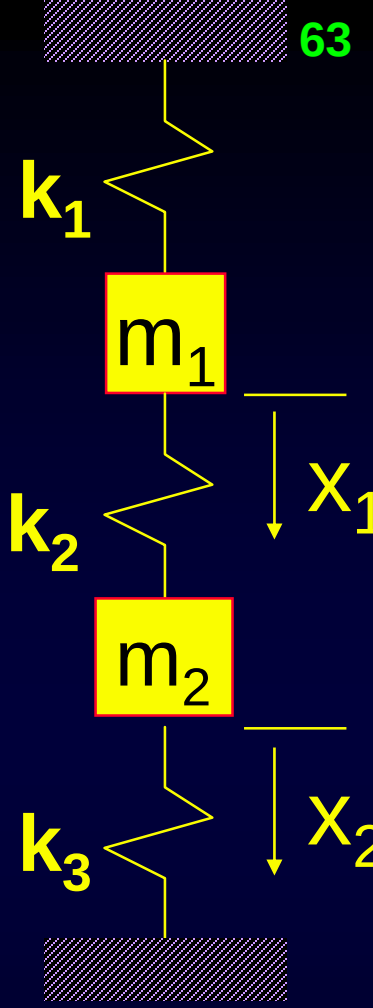
$$= 2k/m \pm k/m$$

$$\omega_{p1}^2 = 2k/m - k/m = k/m$$

$$\omega_{p2}^2 = 2k/m + k/m = 3k/m$$

Thus, there are two natural frequencies, as expected for two degrees of freedom.

Example: Mode Shapes



$$X_F^{(1)} = \frac{X_{2m}^{(1)}}{X_{1m}^{(1)}} = \frac{k_1 + k_2 - m_1\omega_{p1}^2}{k_2} = \frac{2k - m(k/m)}{k} = 1$$

$$= \frac{k_2}{k_2 + k_3 - m_2\omega_{p1}^2} = \frac{k}{2k - m(k/m)} = 1$$

$$X_F^{(2)} = \frac{X_{2m}^{(2)}}{X_{1m}^{(2)}} = \frac{k_1 + k_2 - m_1\omega_{p2}^2}{k_2} = \frac{2k - m(3k/m)}{k} = -1$$

$$= \frac{k_2}{k_2 + k_3 - m_2\omega_{p2}^2} = \frac{k}{2k - m(3k/m)} = -1$$

Time Response

In general, the response is a combination of oscillation at ω_{p1} and at ω_{p2}. The complete time response is now:

$$x_1(t) = A_1 \sin(\omega_{p1} t + \phi_1) X_{1m}^{(1)} + A_2 \sin(\omega_{p2} t + \phi_2) X_{1m}^{(2)}$$

$$x_2(t) = A_1 \sin(\omega_{p1} t + \phi_1) X_{2m}^{(1)} + A_2 \sin(\omega_{p2} t + \phi_2) X_{2m}^{(2)}$$

Or, in matrix form:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_1 \sin(\omega_{p1} t + \phi_1) \begin{bmatrix} X_{1m}^{(1)} \\ X_{2m}^{(1)} \end{bmatrix} + A_2 \sin(\omega_{p2} t + \phi_2) \begin{bmatrix} X_{1m}^{(2)} \\ X_{2m}^{(2)} \end{bmatrix}$$

Example: Time Response

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_1 \sin(\omega_{p1} t + \phi_1) \begin{bmatrix} X_{1m}^{(1)} \\ X_{2m}^{(1)} \end{bmatrix} + A_2 \sin(\omega_{p2} t + \phi_2) \begin{bmatrix} X_{1m}^{(2)} \\ X_{2m}^{(2)} \end{bmatrix}$$

- Solve for unknowns (A_1, ϕ_1, A_2, ϕ_2) from I.C.s
- Note that we don't know the value of $X_{1m}^{(\,)}$ or $X_{2m}^{(\,)}$ only their ratio (from $X_F^{(1)}$ and $X_F^{(2)}$). Therefore, assume a value of 1 for X_{1m} , and let X_{2m} have the value of the mode, $\mathbf{X_F}^{(\,)}$. A_1 and A_2 account for whatever adjustment we've made.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_1 \sin(\omega_{p1} t + \phi_1) \begin{bmatrix} 1 \\ X_F^{(1)} \end{bmatrix} + A_2 \sin(\omega_{p2} t + \phi_2) \begin{bmatrix} 1 \\ X_F^{(2)} \end{bmatrix}$$

Time Response of Principal Modes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_1 \sin(\omega_{p1} t + \phi_1) \begin{bmatrix} 1 \\ X_F^{(1)} \end{bmatrix} + A_2 \sin(\omega_{p2} t + \phi_2) \begin{bmatrix} 1 \\ X_F^{(2)} \end{bmatrix}$$

If we have excited a principal mode, then the system vibrates in only that mode.

If we excite only the first principal mode, then the system oscillates at ω_{p1} so the response is:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_1 \sin(\omega_{p1} t + \phi_1) \begin{bmatrix} 1 \\ X_F^{(1)} \end{bmatrix}$$

$$x_1(t) = A_1 \sin(\omega_{p1} t + \phi_1)$$

$$x_2(t) = X_F^{(1)} A_1 \sin(\omega_{p1} t + \phi_1)$$

Time Response

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_1 \sin(\omega_{p1} t + \phi_1) \begin{bmatrix} 1 \\ X_F^{(1)} \end{bmatrix} + A_2 \sin(\omega_{p2} t + \phi_2) \begin{bmatrix} 1 \\ X_F^{(2)} \end{bmatrix}$$

If we excite only one principal mode, then the system oscillates only at the frequency responding to that principal mode:

For the 1st principal mode:

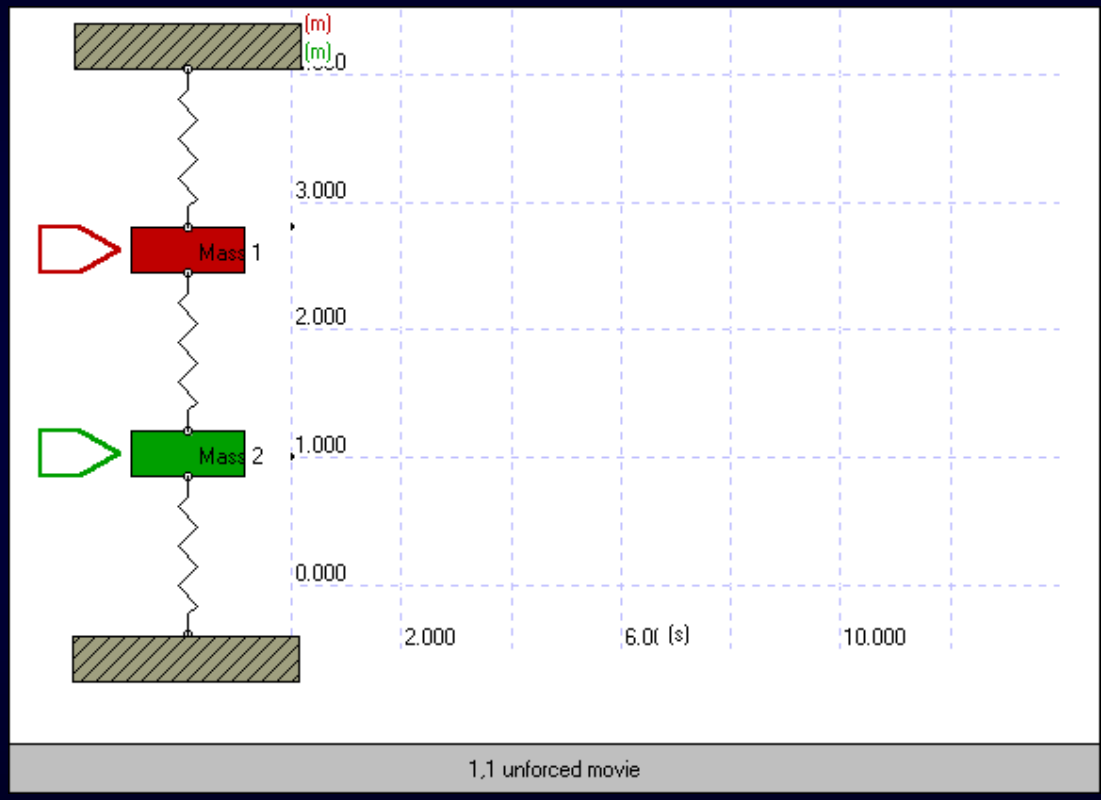
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_1 \sin(\omega_{p1} t + \phi_1) \begin{bmatrix} 1 \\ X_F^{(1)} \end{bmatrix}$$

For the 2nd principal mode:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A_2 \sin(\omega_{p2} t + \phi_2) \begin{bmatrix} 1 \\ X_F^{(2)} \end{bmatrix}$$

1,1 Mode Shape

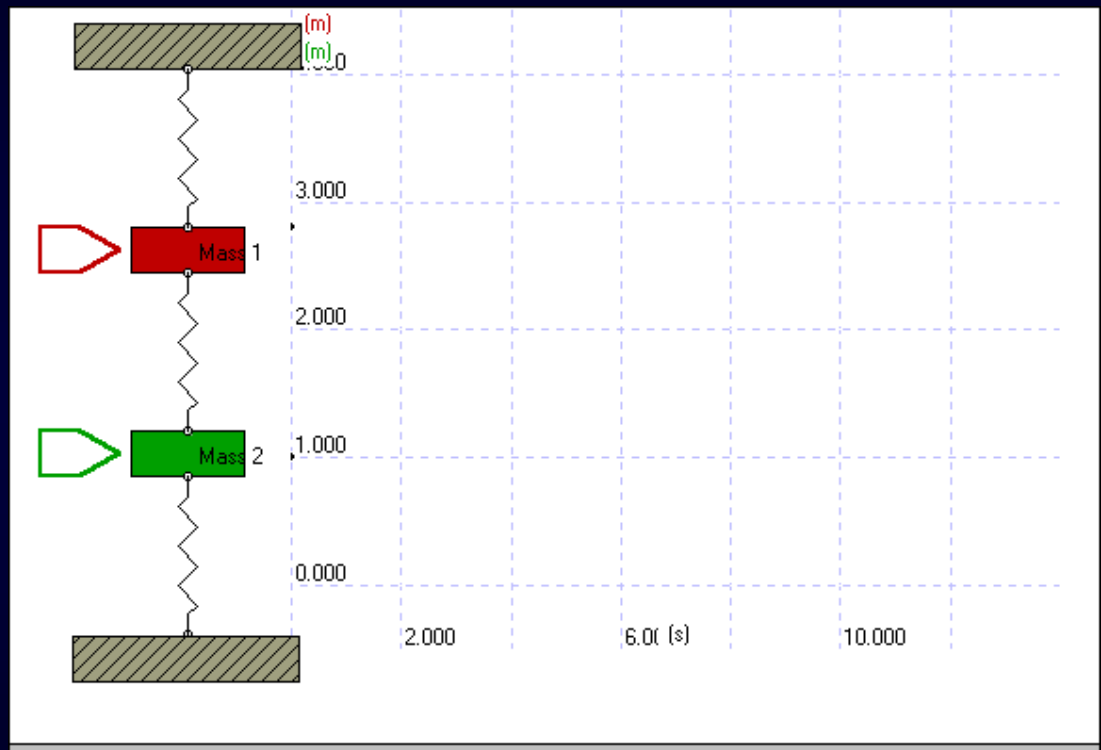
- X_{1m} and X_{2m} are:
- Equal (but don't have to be for a principal mode)
- In-phase
- Note: Pure 1,1 response is created by using only 1,1 initial conditions
- Initial conditions to excite principal modes will be different if system is different.



1,-1 Mode Shape

X_{1m} and X_{2m} are:

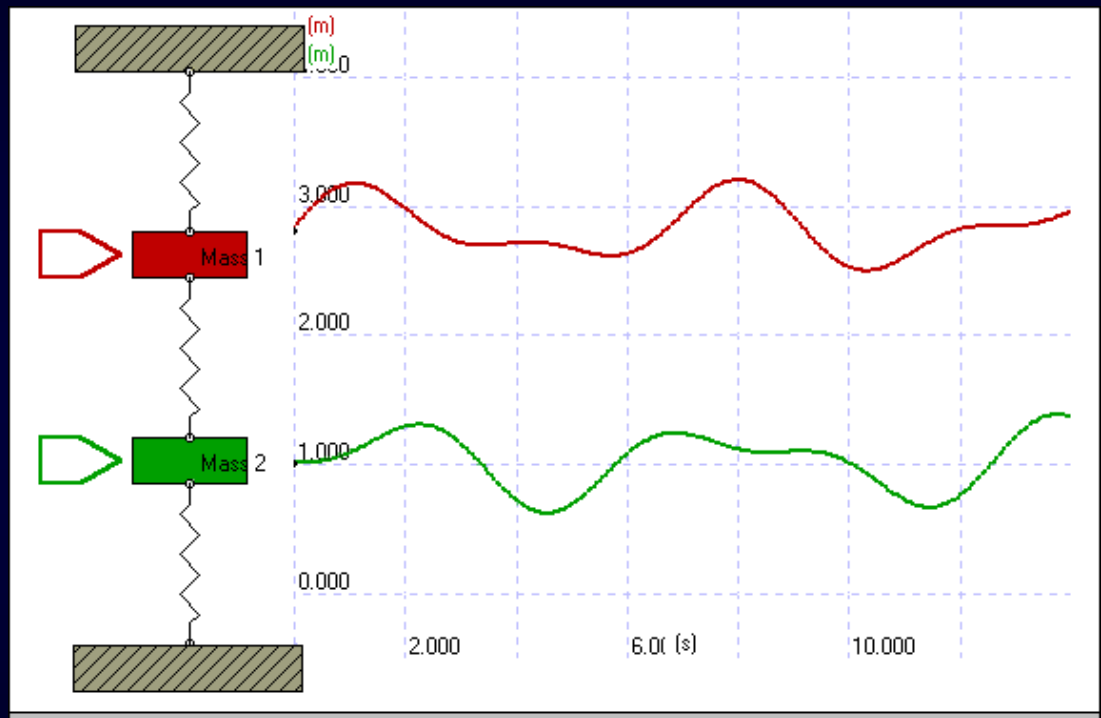
- Equal
- Out-of-phase
  Note: Pure 1,-1 response is created by using 1,-1 initial conditions



1,-1 unforced movie

Combined Mode Shape

- Made up of 1,1 and 1,-1 response
- Exact solution from:
  - 2 ω's
  - 2 mode shapes
  - Initial condition
- Combined response occurs when non-principal initial conditions are used



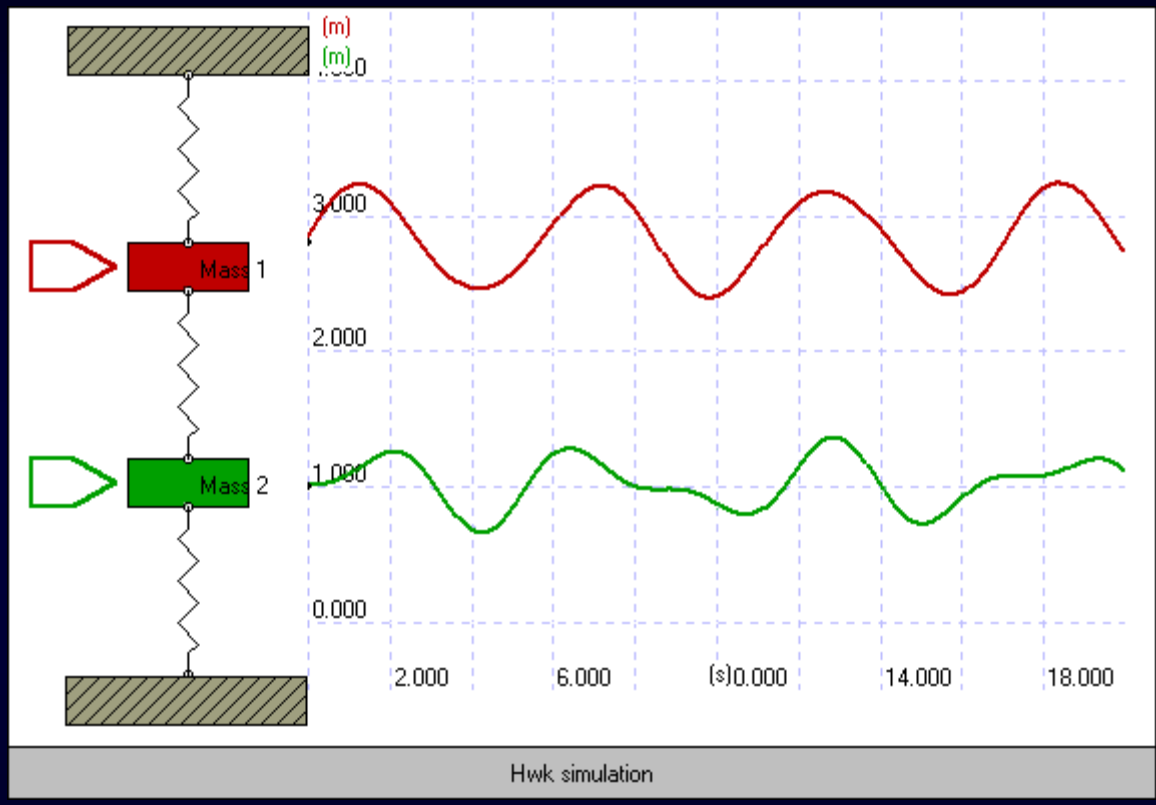
2 DOF combined mode unforced movie

Homework– Due Friday 9/22

- For system we have just modeled, let:
  - $k_1 = 2$ N/m
  - $k_2 = 1$ N/m
  - $k_3 = 2$ N/m
  - $m_1 = 2$ kg
  - $m_2 = 1$ kg

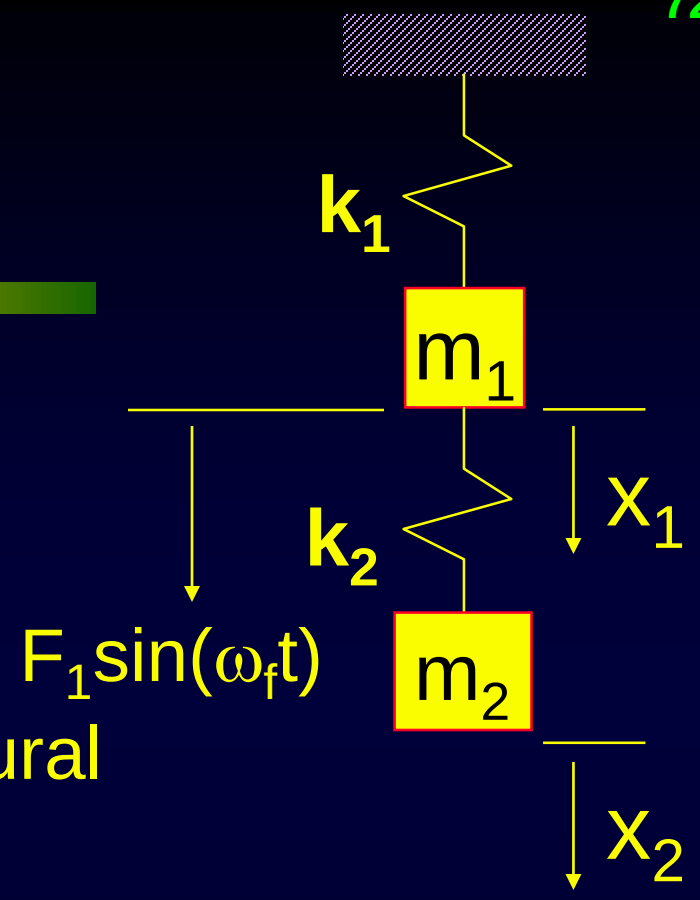
a) Determine the principal natural frequencies

b) Determine the principal mode shapes



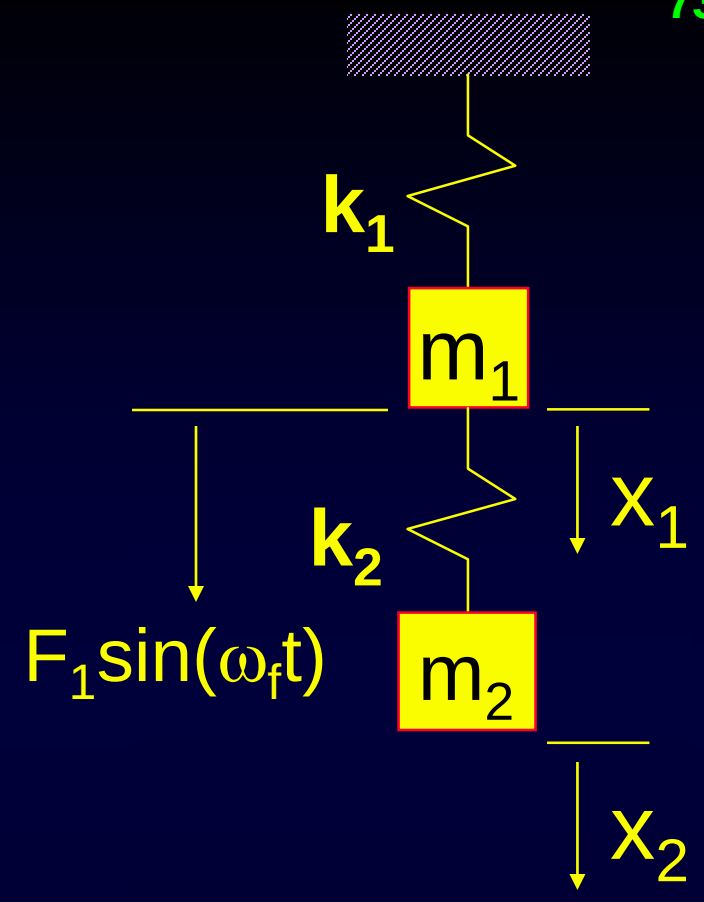
c) Apply an initial velocity on mass 1 of .5 m/s upward. Simulate the response and plot the position of m_1 and m_2 for the first 20 seconds. Plot these on the same graph.

Forced Vibration



- Forced response of a 2 DOF system is similar to that of a single DOF system
- Resonance will occur at the natural frequency of the system
- 2 natural frequencies, so resonance will occur twice, at each natural frequency

Equations of Motion: Forcing Force



$$\sum F = m\ddot{x},$$

Write equation for each DOF:

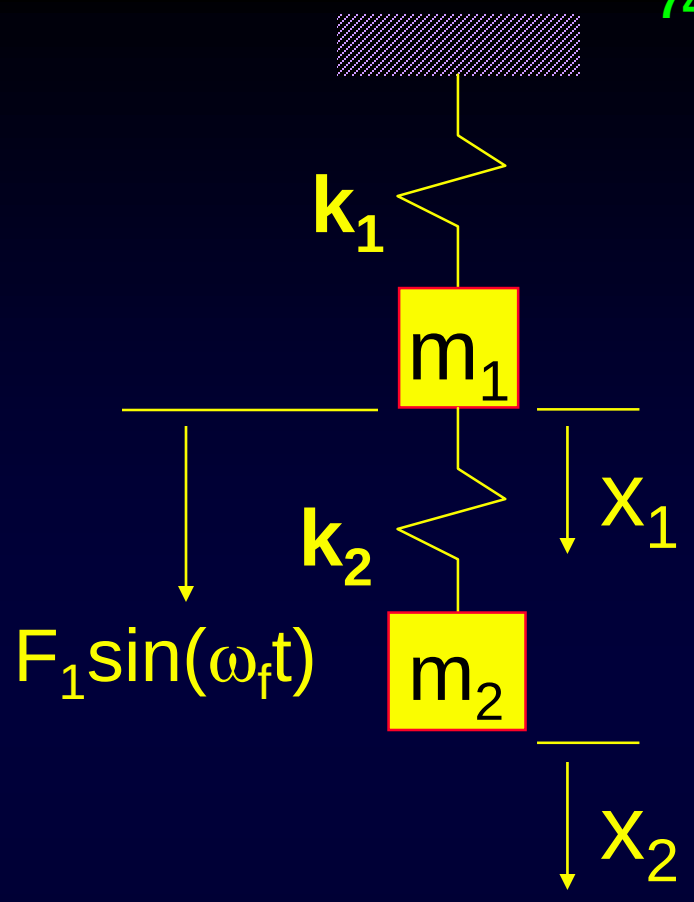
$$-k_1 x_1 + k_2(x_2 - x_1) + F_1 \sin\omega_f t = m_1 \ddot{x}_1$$

$$-k_2(x_2 - x_1) - k_3 x_2 = m_2 \ddot{x}_2$$

Or, in matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \sin\omega_f t \\ 0 \end{Bmatrix}$$

Equations of motion



From our previous work, we know that a solution for $x_1(t)$ is:

$$x_1(t) = X_{1m}\sin(\omega_f t + \phi), \text{ so}$$

$$\ddot{x}_1(t) = -\omega_f^2 X_{1m}\sin(\omega_f t + \phi)$$

So the previous matrices can be combined, as shown on the next slide.

Matrix Formulation

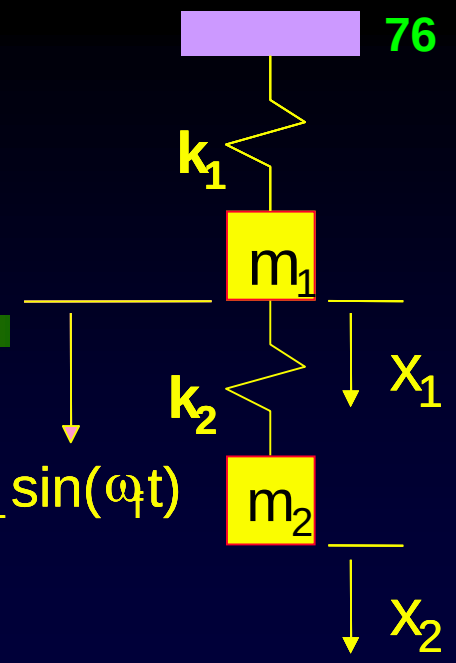
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} -\omega_f^2 X_{1m} \sin \omega_f t \\ -\omega_f^2 X_{2m} \sin \omega_f t \end{Bmatrix} +$$

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} X_{1m} \sin \omega_f t \\ X_{2m} \sin \omega_f t \end{Bmatrix} = \begin{Bmatrix} F_1 \sin \omega_f t \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} -\omega_f^2 X_{1m} \\ -\omega_f^2 X_{2m} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} X_{1m} \\ X_{2m} \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

Simplified Matrix & Determinant

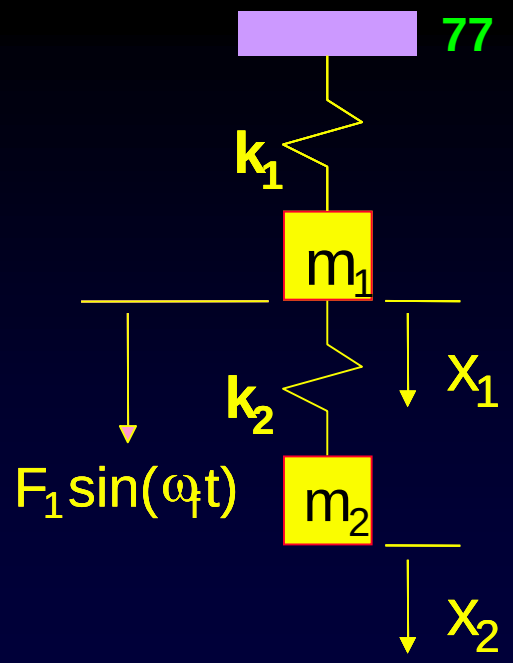


$$\begin{bmatrix} k_1 + k_2 - m_1\omega_f^2 & -k_2 \\ -k_2 & k_2 - m_2\omega_f^2 \end{bmatrix} \begin{Bmatrix} X_{1m} \\ X_{2m} \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

$$\det \begin{bmatrix} k_1 + k_2 - m_1\omega_f^2 & -k_2 \\ -k_2 & k_2 - m_2\omega_f^2 \end{bmatrix} =$$

$$= \left(k_1 + k_2 - m_1\omega_f^2\right)\left(k_2 - m_2\omega_f^2\right) - k_2^2$$

Determinant Simplified



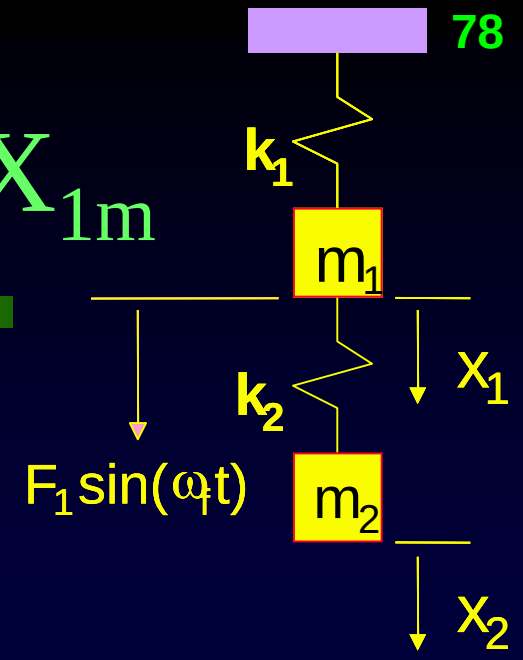
$$= \left(k_1 + k_2 - m_1\omega_f^2\right)\left(k_2 - m_2\omega_f^2\right) - k_2^2$$

which expands to：

$$= m_1 m_2 \omega_f^4 - \omega_f^2\left[m_1 k_2 + m_2\left(k_1 + k_2\right)\right] + k_1 k_2 + k_2^2 - k_2^2$$

$$= m_1 m_2 \omega_f^4 - \omega_f^2\left[m_1 k_2 + m_2\left(k_1 + k_2\right)\right] + k_1 k_2$$

Solution via Cramer's Rule: X_{1m}

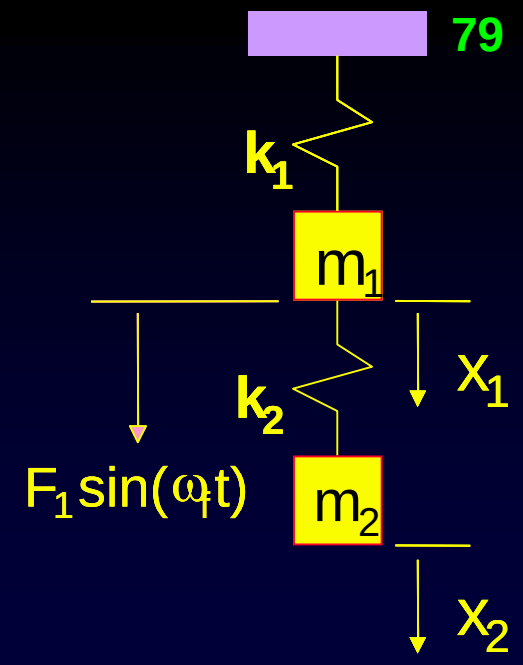


$$X_{1m} = \frac{\begin{vmatrix} F_1 & -k_2 \\ 0 & \left(k_2 - m_2\omega_f^2\right) \end{vmatrix}}{\begin{vmatrix} \left(k_1 + k_2 - m_1\omega_f^2\right) & -k_2 \\ -k_2 & \left(k_2 - m_2\omega_f^2\right) \end{vmatrix}}$$

$$X_{1m} = \frac{F_1\left(k_2 - m_2\omega_f^2\right)}{\left(k_1 + k_2 - m_1\omega_f^2\right)\left(k_2 - m_2\omega_f^2\right) - k_2^2}$$

$$X_{1m} = \frac{F_1 k_2\left(1 - \frac{m_2}{k_2}\omega_f^2\right)}{k_1\left(1 + \frac{k_2}{k_1} - \frac{m_1}{k_1}\omega_f^2\right)k_2\left(1 - \frac{m_2}{k_2}\omega_f^2\right) - k_2^2}$$

"Complification"



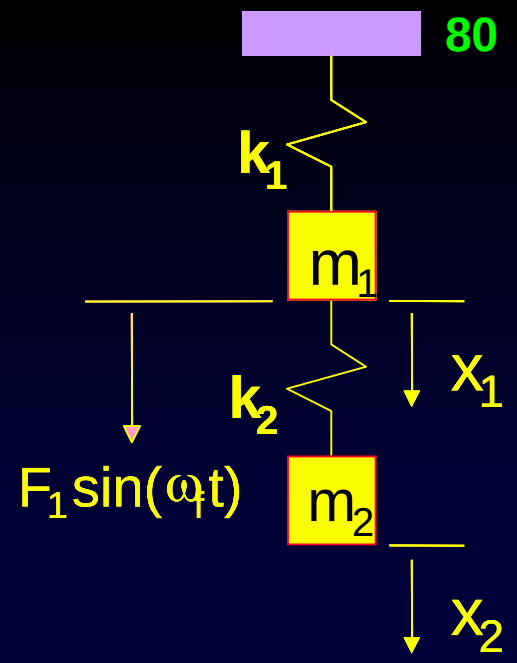
$$X_{1m} = \frac{F_1 k_2 \left(1 - \frac{m_2}{k_2}\omega_f^2\right)}{k_1\left(1 + \frac{k_2}{k_1} - \frac{m_1}{k_1}\omega_f^2\right)k_2\left(1 - \frac{m_2}{k_2}\omega_f^2\right) - k_2^2}$$

$$X_{1m} = \frac{F_1\left(1 - \frac{m_2}{k_2}\omega_f^2\right)}{k_1\left(1 + \frac{k_2}{k_1} - \frac{m_1}{k_1}\omega_f^2\right)\left(1 - \frac{m_2}{k_2}\omega_f^2\right) - k_2}$$

$$X_{1m} = \frac{F_1\left(1 - \frac{\omega_f^2}{\omega_{22}^2}\right)}{k_1\left(1 + \frac{k_2}{k_1} - \frac{\omega_f^2}{\omega_{11}^2}\right)\left(1 - \frac{\omega_f^2}{\omega_{22}^2}\right) - k_2}, \text{ where:}$$

$$\omega_{11} = \sqrt{\frac{k_1}{m_1}}, \quad \omega_{22} = \sqrt{\frac{k_2}{m_2}}$$

"Complification," cont.

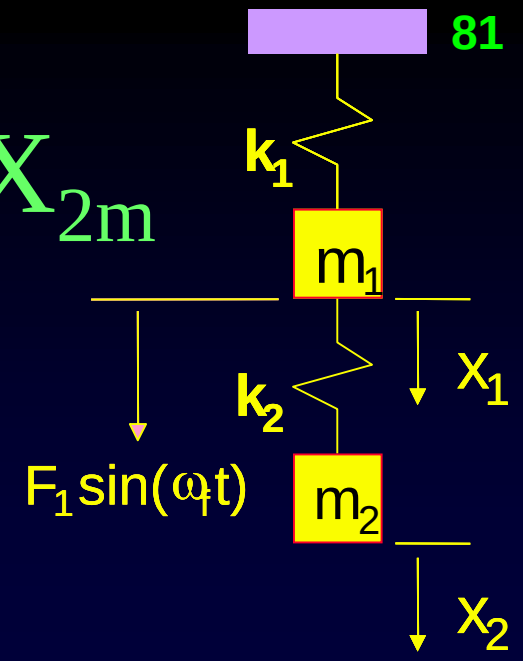


$$X_{1m} = \frac{F_1\left(1-\frac{\omega_f^2}{\omega_{22}^2}\right)}{k_1\left(1+\frac{m_2}{m_1}\frac{k_2/m_2}{k_1/m_1}-\frac{\omega_f^2}{\omega_{11}^2}\right)\left(1-\frac{\omega_f^2}{\omega_{22}^2}\right)-m_2\frac{k_2}{m_2}}$$

$$= \frac{F_1\left(1-\frac{\omega_f^2}{\omega_{22}^2}\right)}{k_1\left(1+\frac{m_2}{m_1}\frac{\omega_{22}^2}{\omega_{11}^2}-\frac{\omega_f^2}{\omega_{11}^2}\right)\left(1-\frac{\omega_f^2}{\omega_{22}^2}\right)-m_2\omega_{22}^2}$$

- Note that X_1 goes to zero when:
  $\omega_f = \omega_{22}$
- This is the basis of a "dynamic vibration absorber"

Solution via Cramer's Rule: X_{2m}

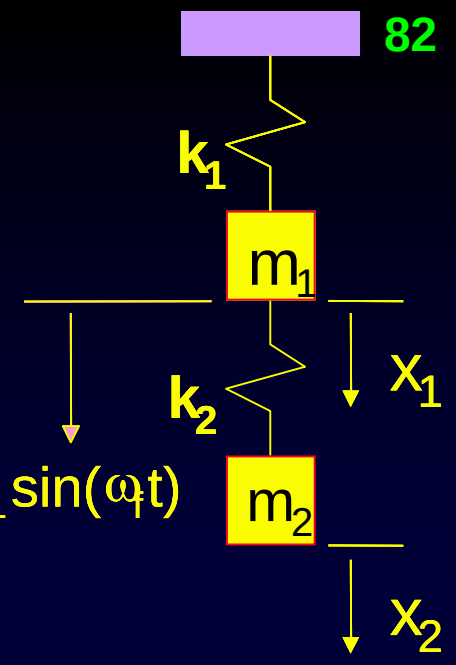


$$X_{2m} = \frac{\begin{vmatrix} \left(k_1 + k_2 - m_2\omega_f^2\right) & F_1 \\ -k_2 & 0 \end{vmatrix}}{\begin{vmatrix} \left(k_1 + k_2 - m_2\omega_f^2\right) & -k_2 \\ -k_2 & \left(k_2 - m_2\omega_f^2\right) \end{vmatrix}}$$

$$X_{2m} = \frac{F_1}{k_1\left(1 + \frac{m_2}{m_1}\frac{\omega_{22}^2}{\omega_{11}^2} - \frac{\omega_f^2}{\omega_{11}^2}\right)\left(1 - \frac{\omega_f^2}{\omega_{22}^2}\right) - m_2^2\omega_{22}^2}$$

Note that X_{2m} never goes to zero

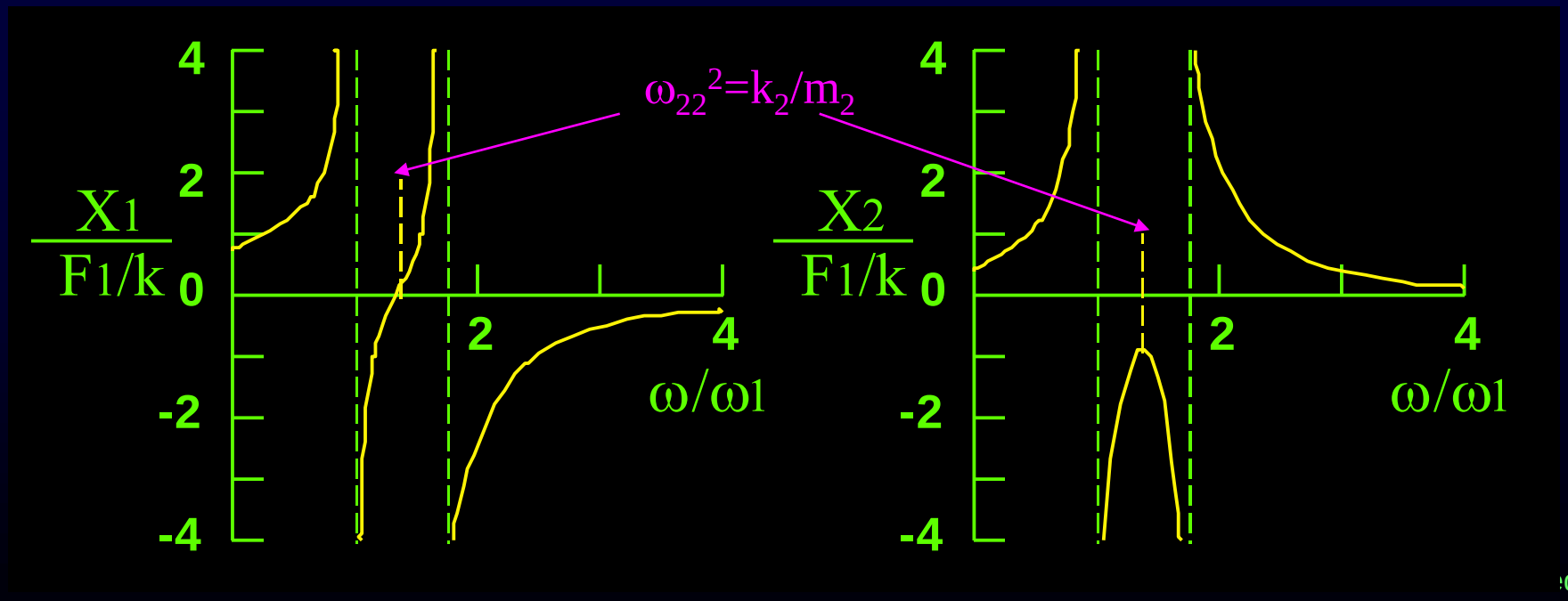
2 DOF Response
Dynamic Absorber Response



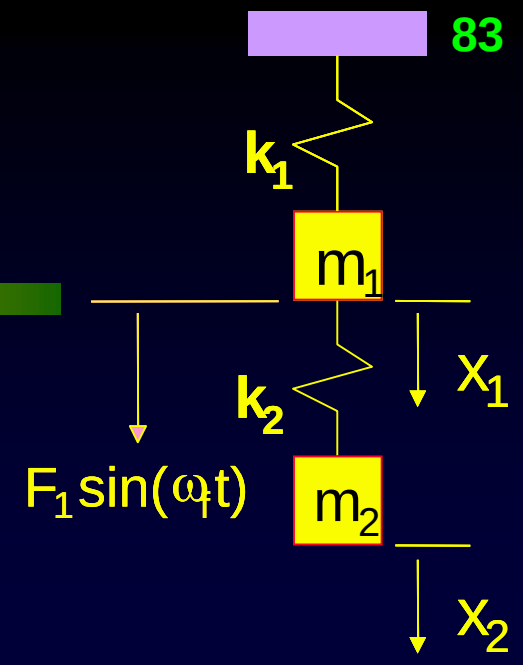
$$X_{1m} = \frac{F_1\left(k_2 - m_2\omega_f^2\right)}{m_1 m_2\left(\omega_f^2 - \omega_{p1}{}^2\right)\left(\omega_f^2 - \omega_{p2}{}^2\right)}$$

$$X_{2m} = \frac{F_1 k_2}{m_1 m_2\left(\omega_f^2 - \omega_{p1}{}^2\right)\left(\omega_f^2 - \omega_{p2}{}^2\right)}$$

Get these equations for X_{1m} & X_{2m} by algebraic manipuation of previous expressions



Comments on Response



- Note that resonance occurs at ω_{p1} and ω_{p2}
- Amplitude is infinite at resonance
- At other frequencies, amplitude can be calculated if the amplitude of the forcing force is known
- Note that there is a frequency at which the vibration of the first mass (to which force is applied) is zero
- This is the basis of the dynamic vibration absorber