LAGRANGE'S EQUATION OF MOTION for SDOF

OBJECTIVE

- To present a general method for obtaining the equation of motion of any linear system – single degree of freedom.

asiri net

GENERAL LAGRANGE EQUATION

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)-\underbrace{\cancel{\frac{\partial T}{\partial x_i}}}_{0}+\frac{\partial V}{\partial x_i}+\frac{\partial D}{\partial \dot{x}_i}=Q_i$$

T : Kinetic energy

***x* is some general arbitrary co-ordinate.**

V : Potential & spring energy

D : Damping dissipation function

Q : External force

i : Co-ordinate number (i = 1 for SDOF systems)

Free-vibration : Q = 0

No damping : D = 0

SDOF

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)+\frac{\partial V}{\partial x}=0$$

Another form of Lagrange's equation using Lagrangian L

For Conservative Systems, Lagrange's equation can be rewritten as

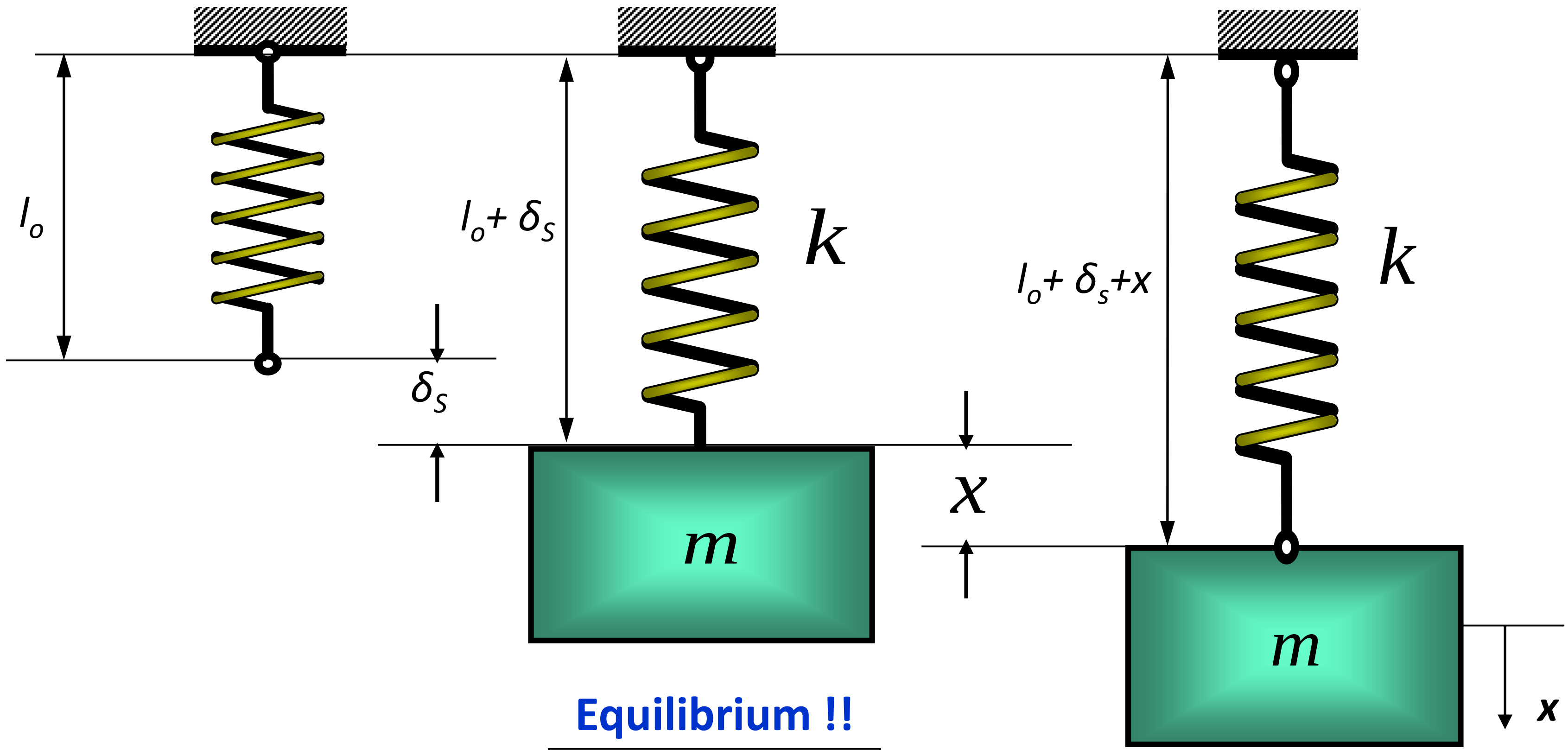
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Where:

$$\boldsymbol{L = T - V}$$

EX.1: Undamped SDOF SYSTEM

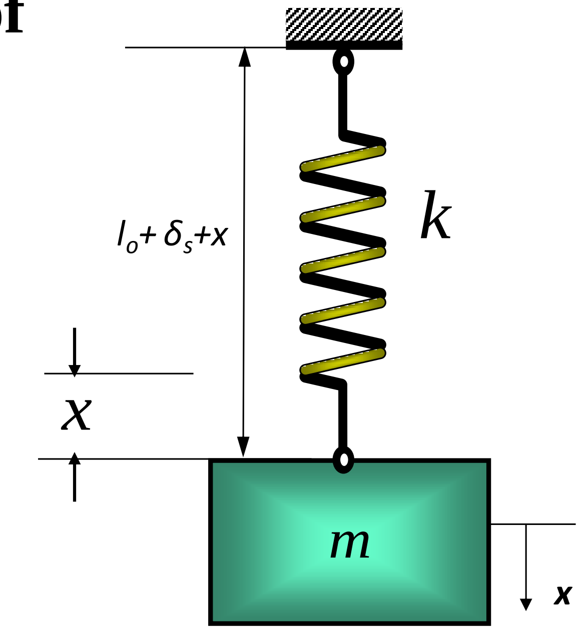
Drive the equation of motion of a simple Spring-Mass system using Lagrange method



Equilibrium !!

$$k\delta_s = mg$$

The given system has only ONE degree of freedom → $q = \{x\}$



(1) Potential Energy

$$\boxed{V = \frac{1}{2}k(x+\delta_s)^2 - mg(x+\delta_s)}$$

(2) Kinetic Energy

$$\boxed{T = \frac{1}{2}m\dot{x}^2}$$

(3) Generalized Forces

No non-conservative forces acting on the system, so the system is "conservative" → $\boxed{Q_{inc} = 0}$

(4) Formulate the "Lagrangian"

$$\boldsymbol{L = T - V}$$

$$\boxed{L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x+\delta_s)^2 + mg(x+\delta_s)}$$

(5) Apply "Lagrange's Equation"

For Conservative systems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \Rightarrow \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}}$$

$$\boxed{\frac{\partial L}{\partial x} = -k(x+\delta_s) + mg}$$

(5) Apply "Lagrange's Equation"

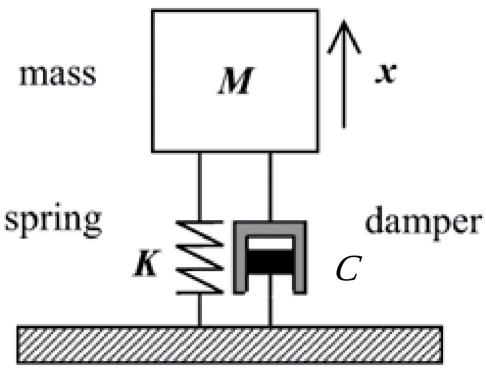
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \Rightarrow m\ddot{x} + k(x + \delta_s) - mg = 0$$

$$\Rightarrow m\ddot{x} + kx + \cancel{k\delta_s} - \cancel{mg} = 0$$

System equation of motion:

$$\boxed{m\ddot{x} + kx = 0}$$

EX.2: Damped SDOF SYSTEM



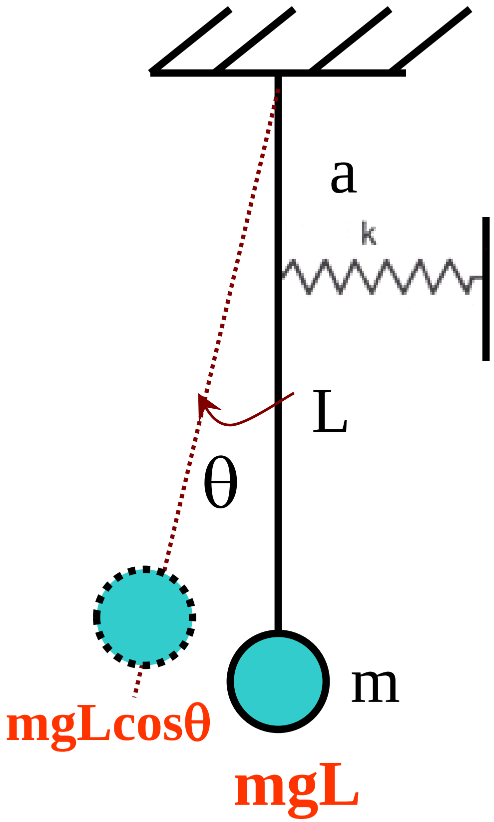
FIND THE EQUATION OF MOTION

$$T=\frac{1}{2}m\dot{x}^2 \rightarrow \frac{\partial T}{\partial \dot{x}}=m\dot{x} \rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)=\frac{d}{dt}(m\dot{x})=m\ddot{x}$$

$$V=\frac{1}{2}kx^2 \rightarrow \frac{\partial V}{\partial x}=kx \qquad D=\frac{1}{2}c\dot{x}^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)+\frac{\partial D}{\partial \dot{x}}+\frac{\partial V}{\partial x}=0 \rightarrow m\ddot{x}+c\dot{x}+kx=0$$

EX.3: PENDULUM WITH SPRING



$$T = \frac{1}{2} I \dot{\theta}^2$$

$$D = 0$$

$$V = \Delta SE + \Delta PE \quad \text{where} \quad \Delta : \text{Change}$$

$$SE : \text{Spring energy, PE : potential energy}$$

$$\text{SE} = \frac{1}{2} k y^2 \quad \text{where} \;\; \text{y} = a\theta$$

$$PE = mgL(1 - \cos\theta)$$

$$V = \frac{1}{2} k (a\theta)^2 + mgL(1 - \cos\theta)$$

OBTAIN DERIVATIVES

$$T = \frac{1}{2} I\dot{\theta}^2$$

$$\rightarrow \frac{\partial T}{\partial \dot{\theta}} = I\dot{\theta} \rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}(I\dot{\theta}) = I\ddot{\theta}$$

$$V = \frac{1}{2} k(a\theta)^2 + mgL(1-\cos\theta)$$

$$\rightarrow \frac{\partial V}{\partial \theta} = ka^2\theta + mgL\sin\theta \approx (ka^2 + mgL)\theta$$

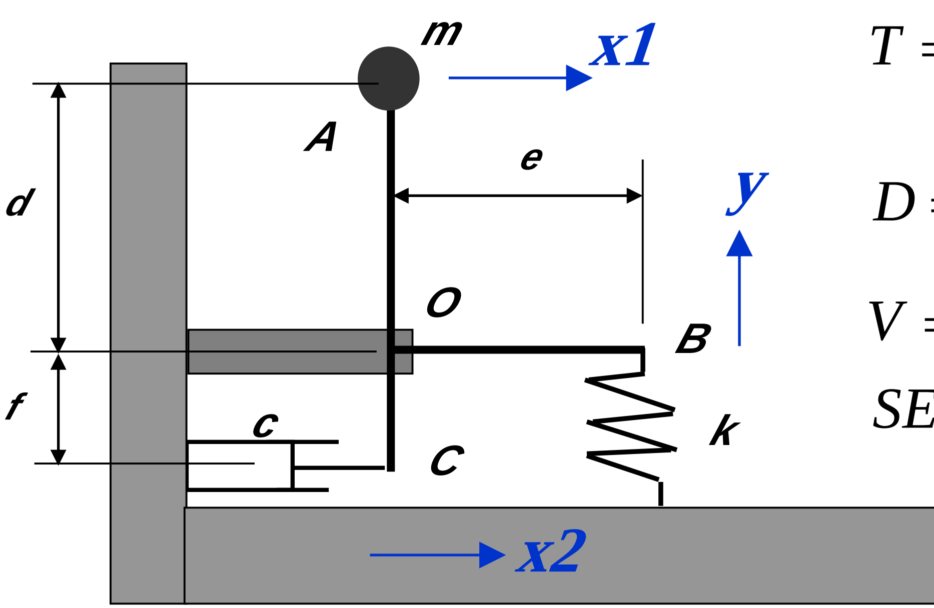
INSERT INTO LAGRANGE EQUATION

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right) + \frac{\partial V}{\partial \theta_i} = 0$$

$$I\ddot{\theta} + (ka^2 + mgL)\theta = 0$$

$$\omega_n = \sqrt{\frac{(ka^2 + mgL)}{I}}$$

EX.4: SYSTEM WITH MASSLESS ARMS



$$T = \frac{1}{2} m \dot{x}_1^{\,2} \quad \text{where } x_1/d = \theta$$

$$D = \frac{1}{2} c \dot{x}_2^{\,2} \quad \text{where } x_2/f = \theta$$

$$V = \Delta SE + \Delta PE \quad \text{where } \Delta : \text{Change}$$

SE : Spring energy, PE : potential energy

$$\text{SE} = \frac{1}{2} k y^2 \quad \text{where } y/e = \theta$$

$$PE = Final - Initial = mgd\cos\theta - mgd = mgd(\cos\theta - 1)$$

$$V = \frac{1}{2} k (e\theta)^2 + mgd(\cos\theta - 1)$$

OBTAIN DERIVATIVES

$$T = \frac{1}{2} m(d\dot{\theta})^2$$

$$\rightarrow \frac{\partial T}{\partial \dot{\theta}} = md^2\dot{\theta} \rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}(md^2\dot{\theta}) = md^2\ddot{\theta}$$

$$V = \frac{1}{2} k(e\theta)^2 + mgd(\cos\theta - 1)$$

$$\rightarrow \frac{\partial V}{\partial \theta} = ke^2\theta - mgd\sin\theta \approx (ke^2 - mgd)\theta$$

$$D = \frac{1}{2} c(f\dot{\theta})^2$$

$$\rightarrow \frac{\partial D}{\partial \dot{\theta}} = cf^2\dot{\theta}$$

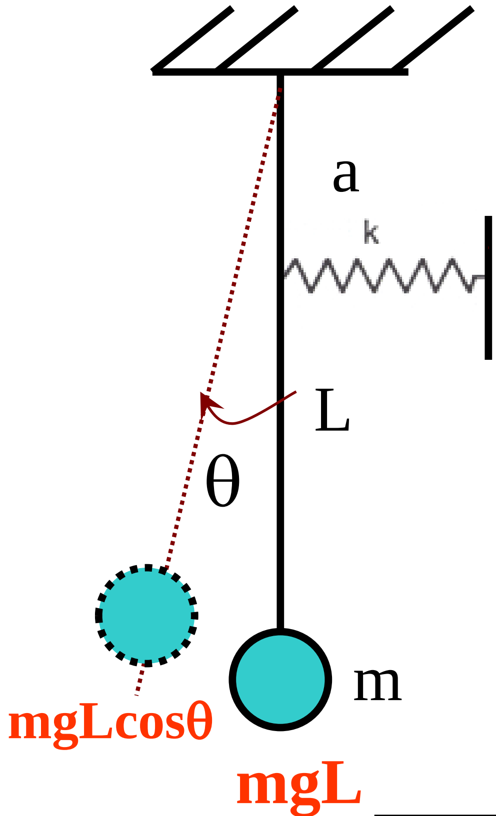
INSERT INTO LAGRANGE EQUATION

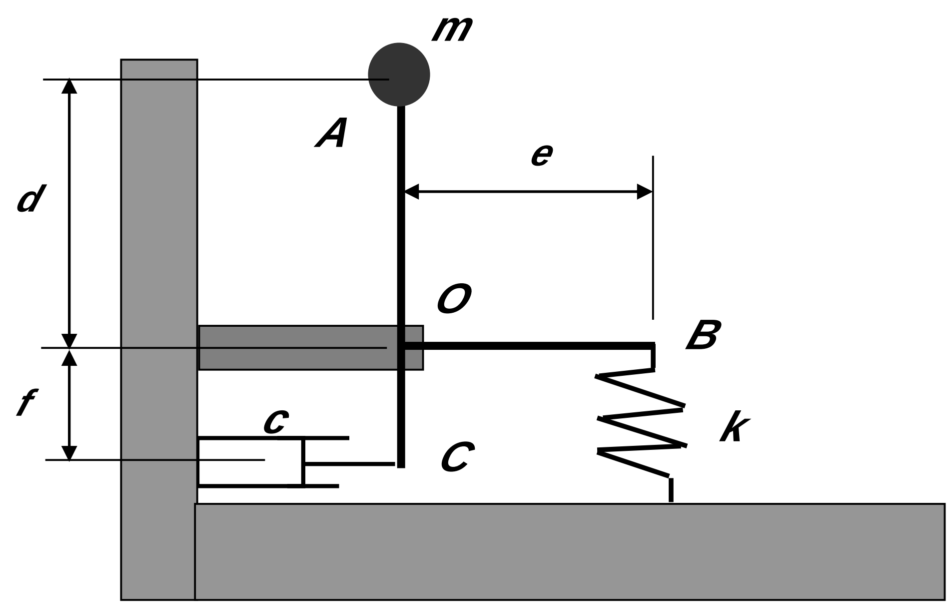
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right)+\frac{\partial V}{\partial \theta_i}+\frac{\partial D}{\partial \dot{\theta}_i}=0$$

$$md^2\ddot{\theta}+(ke^2-mgd)\theta+cf^2\dot{\theta}=0$$

$$\omega_n=\sqrt{\frac{(ke^2-mgd)}{md^2}}$$

WHAT IS THE DIFFERENCE?





$$\omega_n = \sqrt{\frac{(ka^2 + mgL)}{I}}$$

$$\omega_n = \sqrt{\frac{(ke^2 - mgd)}{md^2}}$$