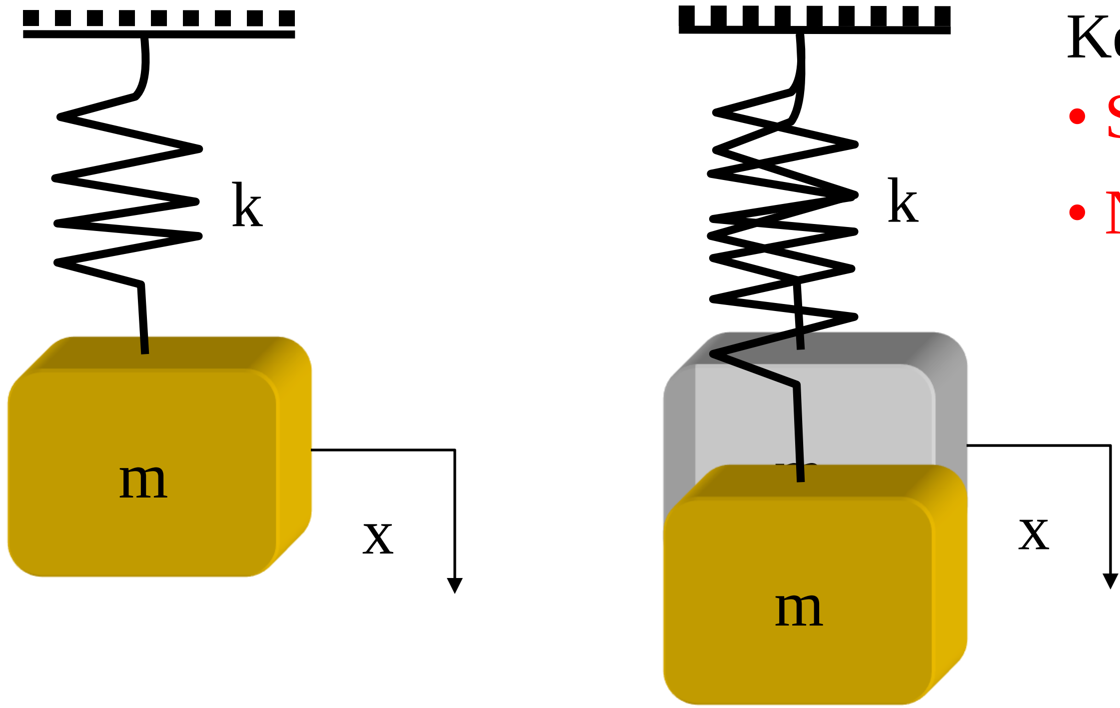
Single Degree of Freedom Free Vibration

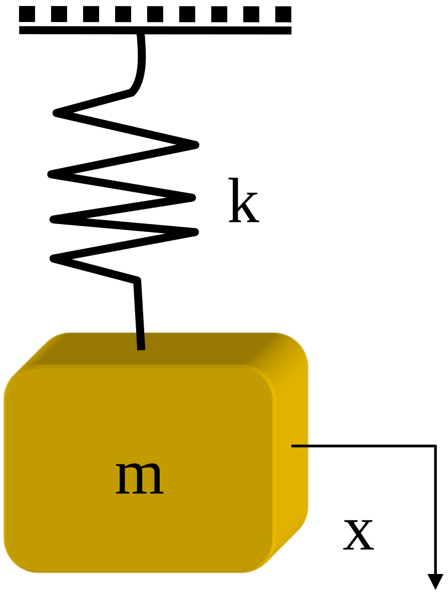


Key Points:

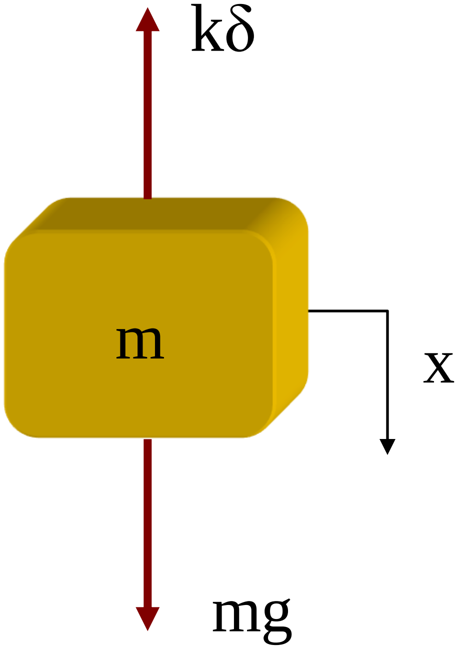
- System is un-damped
- No external forces

Given some initial conditions, Determine the resulting motion

Free Body Diagram: Static Case

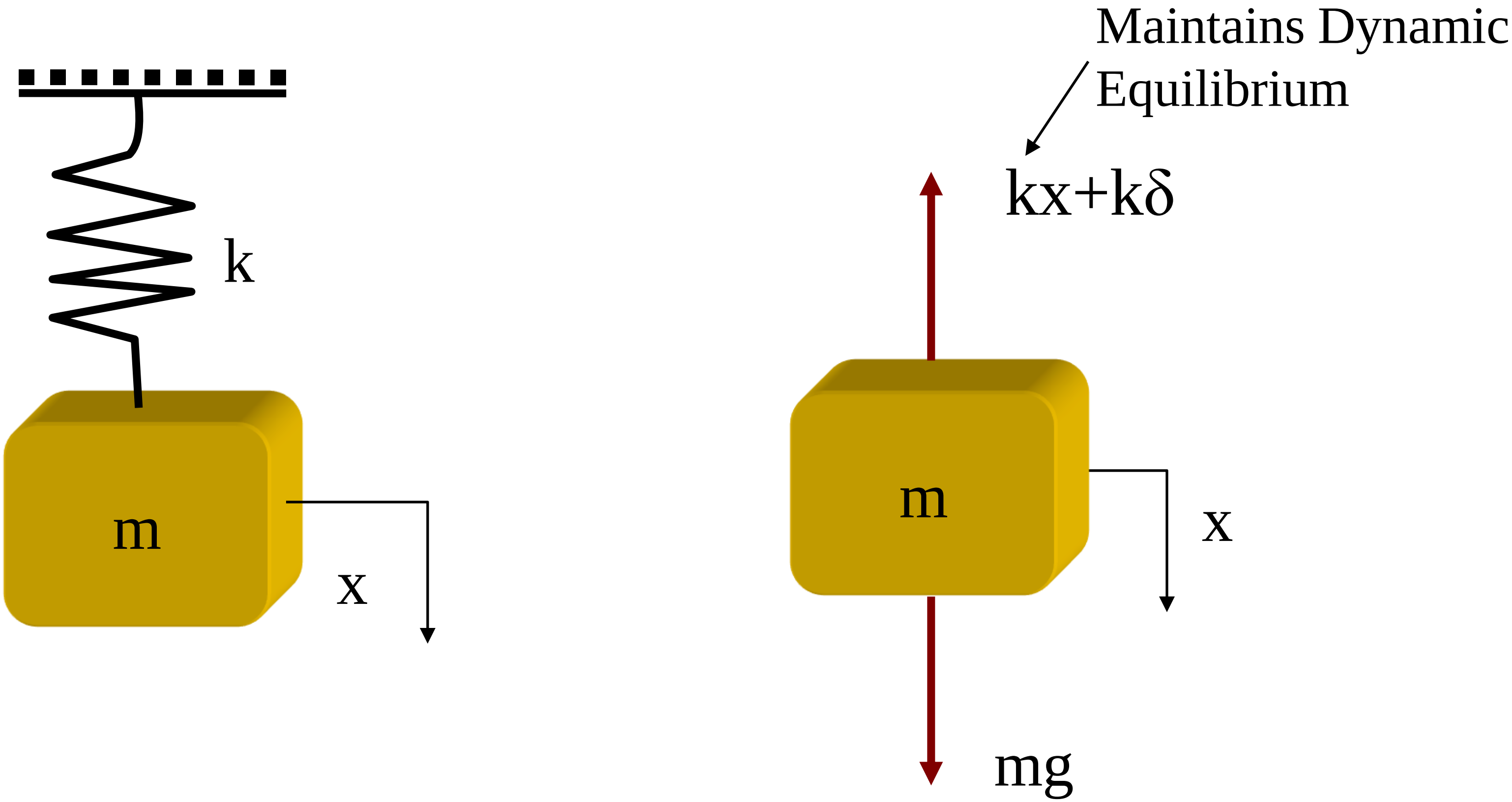


At rest, x = 0 (static equilibrium)



$$mg = k\delta$$

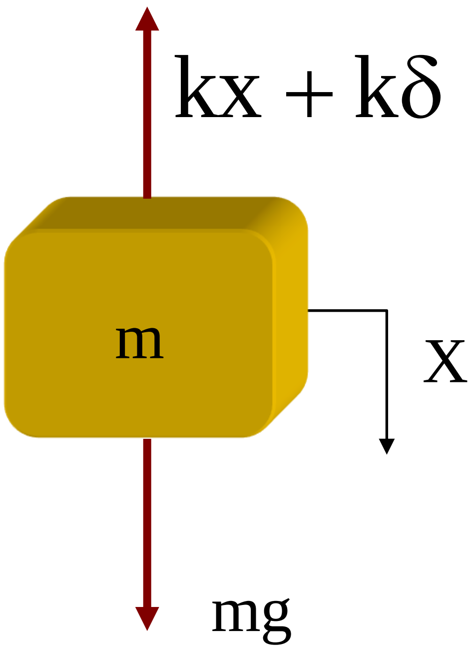
Free Body Diagram: Dynamic Equilibrium



Note: x is measured from the static equilibrium position.

Free Body Diagram: Dynamic Case

Free Body Diagram



Apply Newton's 2nd Law

$$\sum F = m\ddot{x}$$

$$\sum F_{x\downarrow +} = m\ddot{x}$$

$$mg - (kx + k\delta) = m\ddot{x}$$

Equation of motion (EOM) $$m\ddot{x} + kx = 0$$

Equation of motion

$$m\ddot{x} + kx = 0$$

2nd order Differential equation

homogeneous

linear

constant coefficients

Forms of solution:

$x(t)=X \sin(\omega t + \Phi)$

$x(t)=X \cos(\omega t - \Phi)$

$x(t)=Ce^{st}$

Equation of motion $$m\ddot{x} + kx = 0$$

Assume $x(t) = Ce^{st}$

$$\dot{x}(t) = sCe^{st}$$

$$\ddot{x}(t) = s^2Ce^{st}$$

$$ms^2Ce^{st} + kCe^{st} = 0$$

$$(ms^2 + k)Ce^{st} = 0$$

for a non-trivial solution

$$ms^2 + k = 0$$

Equation of motion $m\ddot{x} + kx = 0$

$$ms^2 + k = 0 \qquad s_{1,2} = \pm j\sqrt{\frac{k}{m}}$$

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

$$x(t) = C_1 e^{j\sqrt{\frac{k}{m}}t} + C_2 e^{-j\sqrt{\frac{k}{m}}t}$$

C_1 and C_2 are arbitrary constants to be determined from initial conditions

$$x(t) = C_1 e^{j\sqrt{\frac{k}{m}}t} + C_2 e^{-j\sqrt{\frac{k}{m}}t}$$

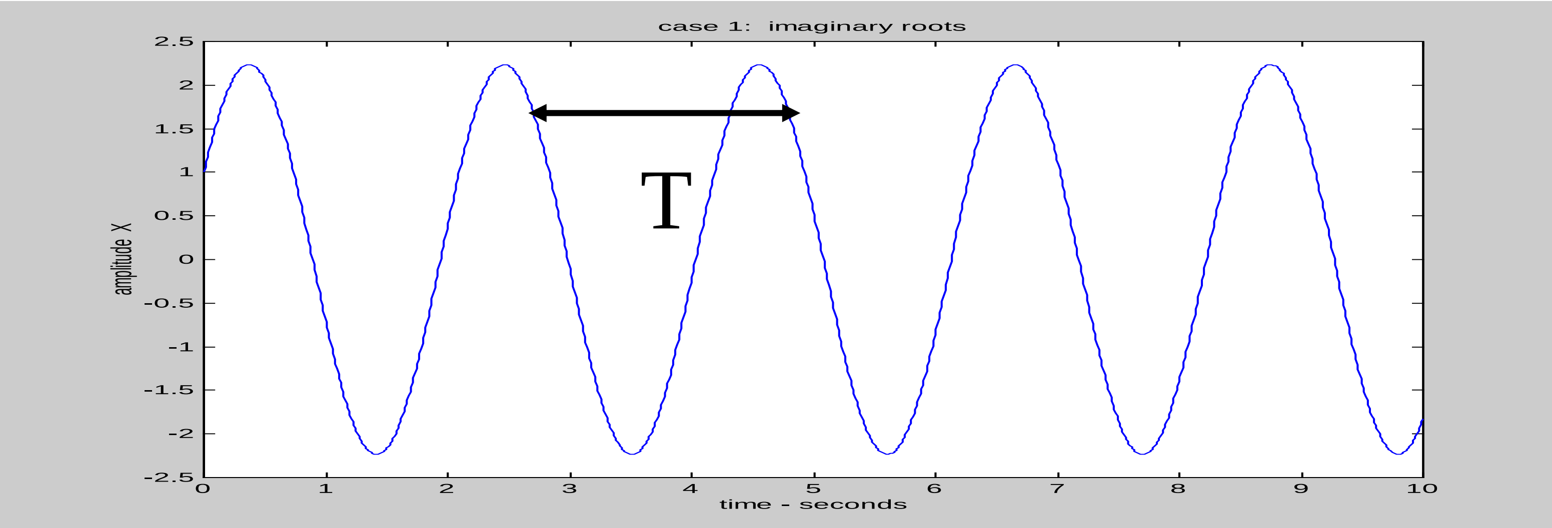
Recall Euler's identity: $e^{\pm j\theta} = \cos(\theta) \pm j\sin(\theta)$

$$x(t) = C_1\left(\cos\sqrt{\frac{k}{m}}t + j\sin\sqrt{\frac{k}{m}}t\right) + C_2\left(\cos\sqrt{\frac{k}{m}}t - j\sin\sqrt{\frac{k}{m}}t\right)$$

$$x(t) = (C_1 + C_2)\cos\sqrt{\frac{k}{m}}t + j(C_1 - C_2)\sin\sqrt{\frac{k}{m}}t$$

$$x(t) = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t$$

$$x(t) = A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t$$



$$\omega_n = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} = \text{ natural frequency (rad/sec)}$$

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

$$f_n = \frac{1}{T} = \text{natural frequency}\left(\frac{\text{cycles}}{\text{sec}}, \text{or Hz}\right)$$

$$x(t) = A\cos(2\pi f_n t) + B\sin(2\pi f_n t)$$

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

$$x(0)$$

$$\dot{x}(0)$$

A and B are determined from the **Initial Conditions**

$$x(0) = A\cos(0) + B\sin(0) \quad \cos(0) \to 1, \; \sin(0) \to 0$$

$$X(0) = A$$

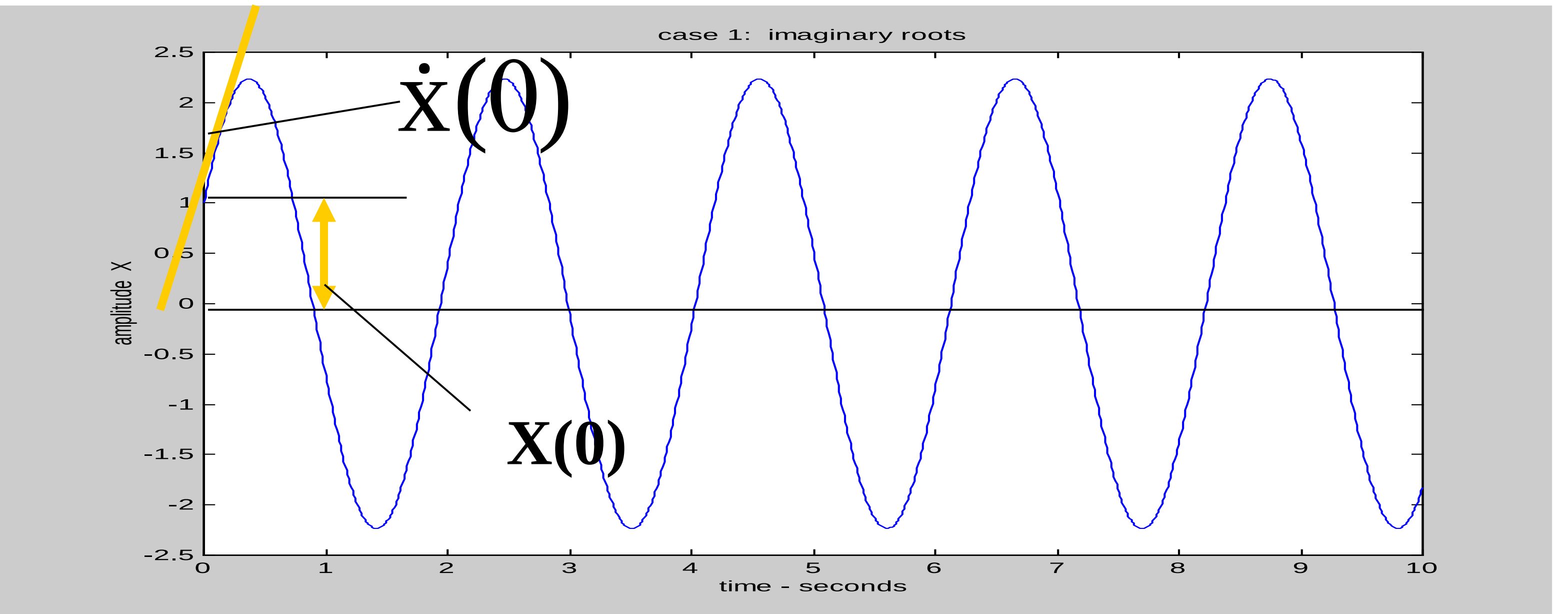
$$\dot{x}(t) = -\omega_n A\sin(\omega_n t) + \omega_n B\cos(\omega_n t)$$

$$\dot{x}(0) = -\omega_n A(0) + \omega_n B(1)$$

$$\frac{\dot{x}(0)}{\omega_n} = B$$

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

$$x(t) = x(0)\cos(\omega_n t) + \frac{\dot{x}(0)}{\omega_n}\sin(\omega_n t)$$



Harmonic Motion

$$x(t) = X\cos\left(\omega_n t - \phi\right)$$

$$X = \sqrt{A^2 + B^2} = \sqrt{x(0)^2 + \left(\frac{\dot{x}(0)}{\omega_n}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{\dot{x}(0)}{x(0)\,\omega_n}\right)$$

Amplitude

Phase

Harmonic Motion

$$x(t) = X\sin\left(\omega_n t + \phi\right)$$

$$X = \sqrt{A^2 + B^2} = \sqrt{X(0)^2 + \left(\frac{\dot{X}(0)}{\omega_n}\right)^2} \qquad \phi = \tan^{-1}\left(\frac{x(0)\,\omega_n}{\dot{x}(0)}\right)$$

Amplitude

Phase

Torsional System: SDOF

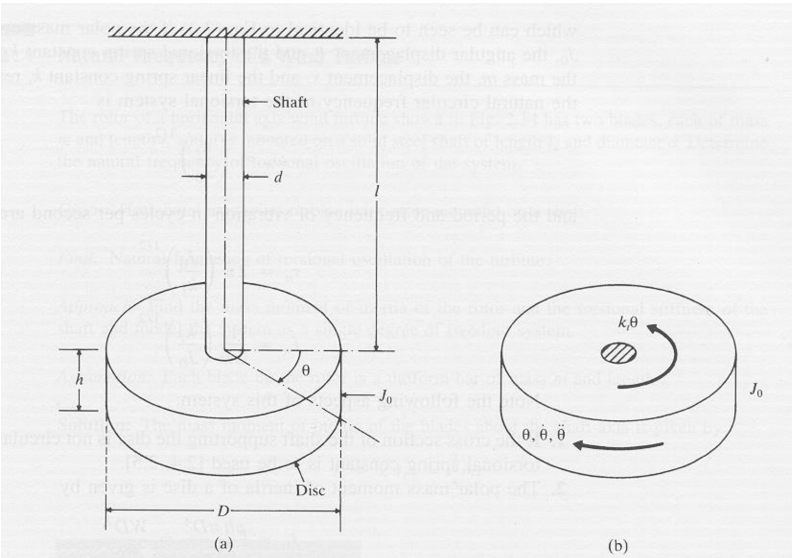


FIGURE 2.13 Torsional vibration of a disc.

Equation of Motion:

$$J_o\ddot{\theta} + k_t\theta = 0$$

Solution Form:

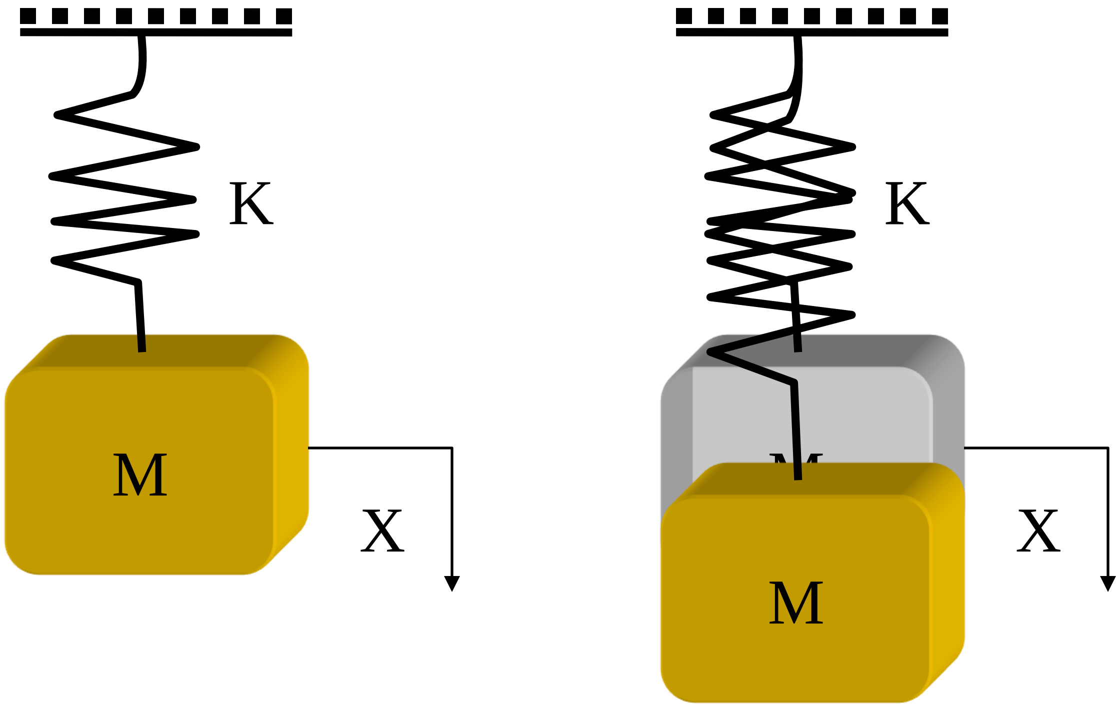
$$\theta(t)=A\cos\left(\omega_n t\right)+B\sin\left(\omega_n t\right)$$

SDOF Translatory System

Equation of Motion: $m\ddot{x} + k\ x = 0$

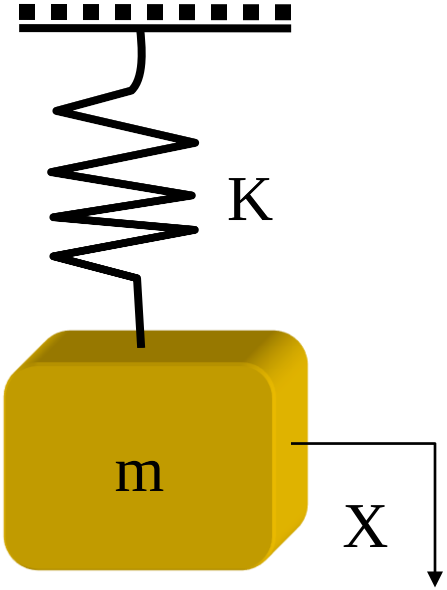
Solution Form: $x(t)=A\cos\left(\omega_n t\right)+B\sin\left(\omega_n t\right)$

Single Degree of Freedom Free Vibration



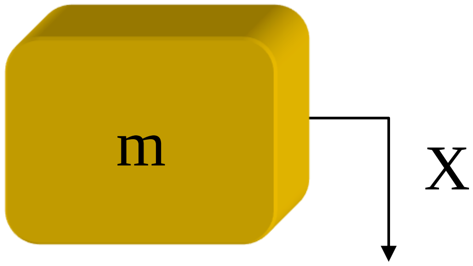
Given an initial condition, Determine the resulting motion

Energy Method: Conservation of Energy



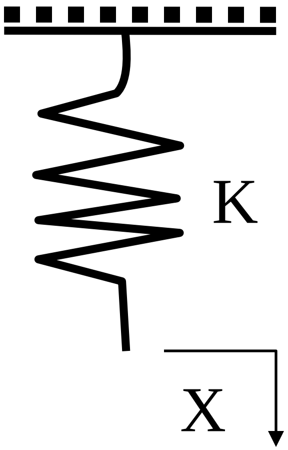
K.E. +P.E. = constant

T + U = constant



$$T = \frac{1}{2} m \dot{X}^2$$

Energy Method: Conservation of Energy



Potential energy = U = all potential energy in the system

U = potential energy in the spring + change in potential energy due to elevation

$$U = mgx + \frac{1}{2}Kx^2 - mgx$$

Energy Method: Conservation of Energy

T + U = constant

$$\frac{d(T+U)}{dt} = 0$$

Therefore the rate of change of system energy must be zero

$$T + U = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

$$\frac{d(T+U)}{dt} = \frac{2}{2} m\dot{x}\ddot{x} + \frac{2}{2} K\dot{x}x = 0$$

Energy Method: Conservation of Energy

$$\frac{d(T+U)}{dt} = \frac{2}{2} m\dot{x}\ddot{x} + \frac{2}{2} Kx\dot{x} = 0$$

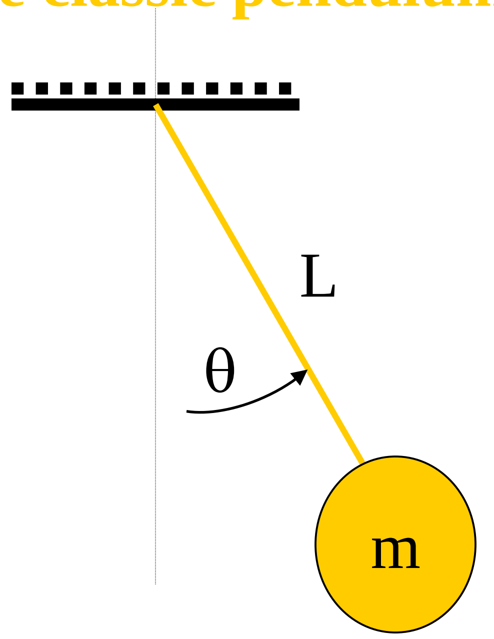
Equation of motion $$m\ddot{x} + kx = 0$$

Solution to equation of motion

$$\omega_n = \sqrt{k/m}$$

$$x(t) = x(0)\cos\omega_n t + \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t$$

The classic pendulum problem



Case 1: Energy method

Determine the equation of motion and the natural frequency

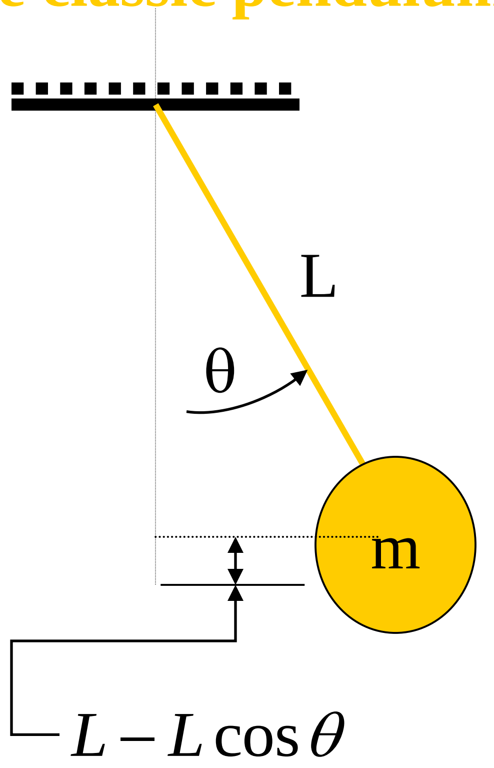
Kinetic Energy

$$T = \frac{1}{2} m \left(\frac{d(\sin\theta)}{dt} L \right)^2$$

for small angles $\sin\theta = \theta$

$$T = \frac{1}{2} m \left(\dot{\theta} L \right)^2$$

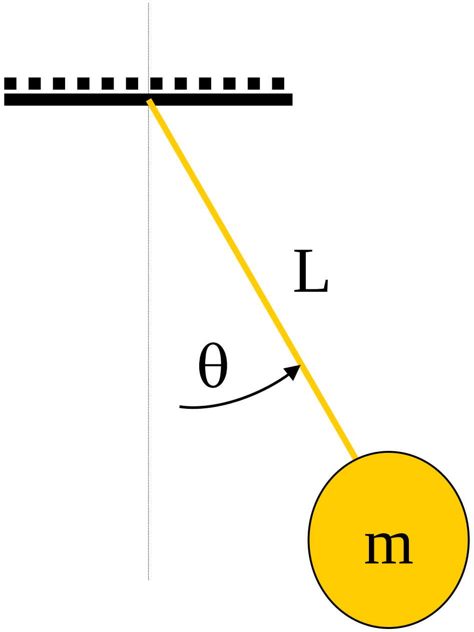
The classic pendulum problem



Potential energy U, is proportional to the change in elevation of the mass

$$U = mg(L - L\cos\theta)$$

The classic pendulum problem



$$T + U = \text{constant}$$

$$T + U = \frac{1}{2} m \left(\dot{\theta} L\right)^2 + mg(L - L\cos\theta)$$

$$\frac{d(T+U)}{dt} = 0$$

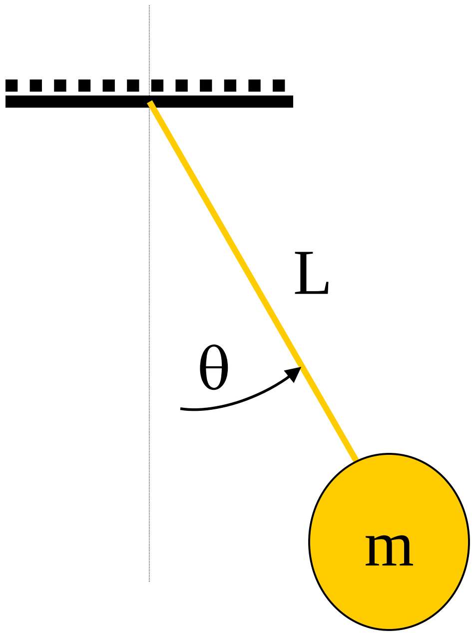
$$\frac{2}{2} m \dot{\theta} L^2 \ddot{\theta} + mgL\sin\theta\dot{\theta} = 0$$

$$\boxed{L\ddot{\theta} + g\theta = 0}$$ Equation of motion

$$\omega_n = \sqrt{g/L}$$

The classic pendulum problem

Case 2: Newton's method

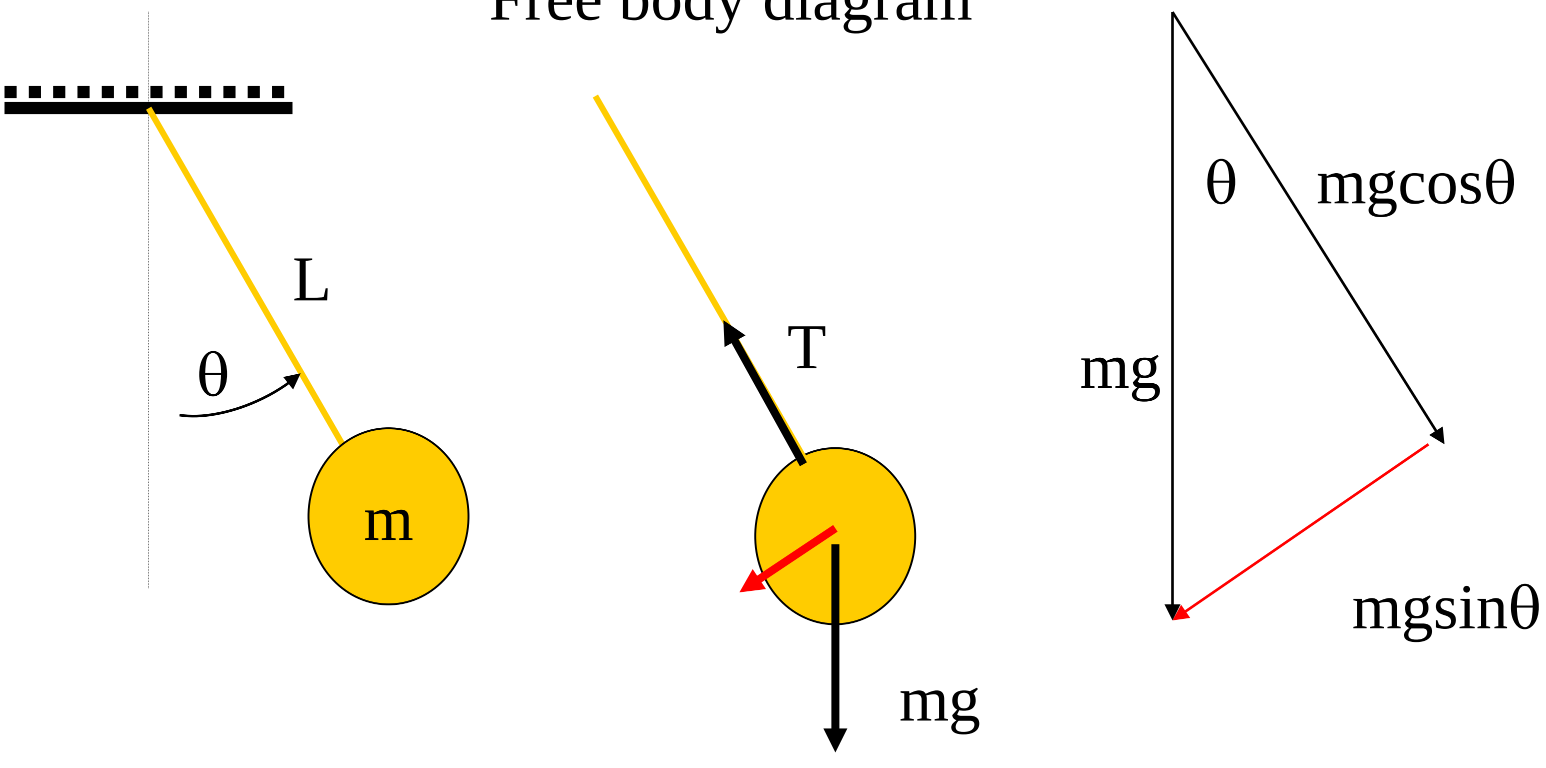


$$\sum F = m\ddot{x}$$

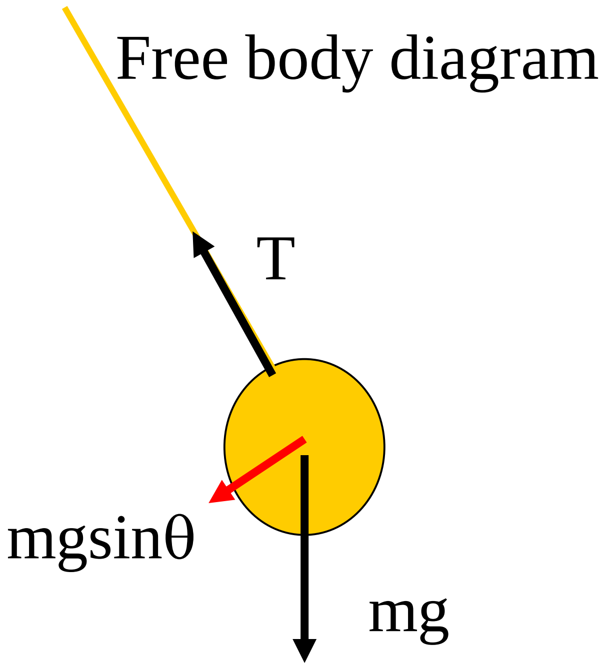
$$\sum M = J\ddot{\theta}$$

The classic pendulum problem

Free body diagram



The classic pendulum problem



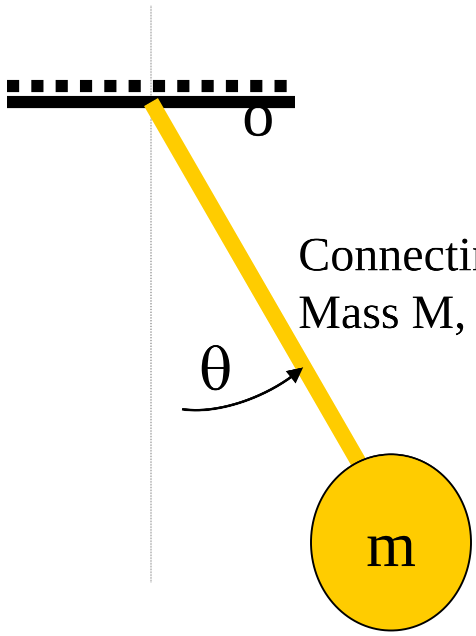
$$\sum M = J\ddot{\theta}$$

$$-mg\sin\theta L = J\ddot{\theta}$$

$$mL^2\ddot{\theta} + mgL\theta = 0$$

$$L\ddot{\theta} + g\theta = 0$$

Pendulum problem: mass of bar is considered



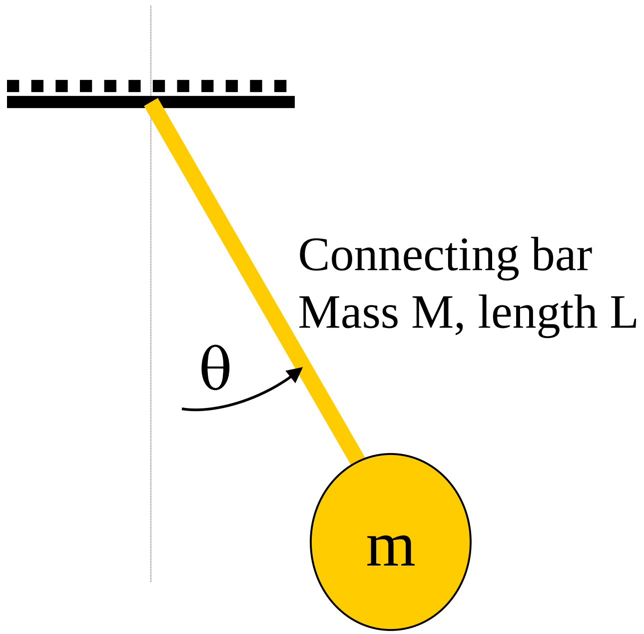
Kinetic Energy

$$T = T_{bar} + T_{bob} = \frac{1}{2} J_o \dot{\theta}^2 + \frac{1}{2} m L^2 \dot{\theta}^2$$

$$J_o = J_{cg} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$$

$$T = \frac{1}{2}\left(m + \frac{M}{3}\right) L^2 \dot{\theta}^2$$

Pendulum problem: mass of bar is considered



Potential Energy

$$U = U_{bar} + U_{bob}$$

$$U = Mg\left(\frac{L}{2} - \frac{L}{2}\cos\theta\right) + mg(L - L\cos\theta)$$

Equation of Motion

$$\frac{d}{dt}\left(T+U\right)=0$$

$$\frac{d}{dt}\left(\frac{1}{2}\left(m+\frac{M}{3}\right)L^2\dot{\theta}^2+Mg\frac{L}{2}(1-\cos\theta)+mgL(1-\cos\theta)\right)=0$$

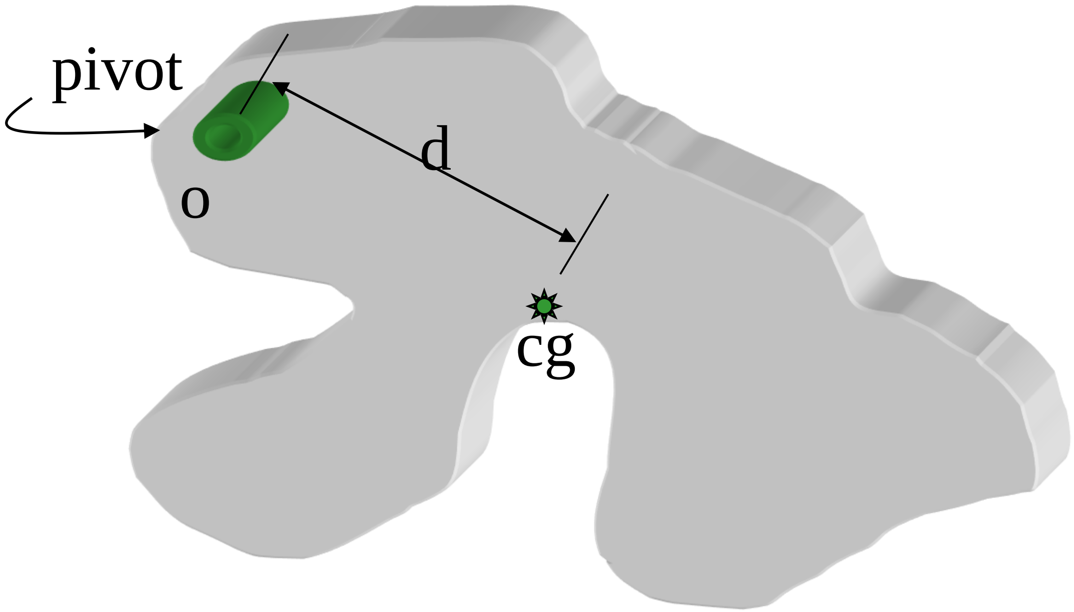
$$\left(m+\frac{M}{3}\right)L^2\ddot{\theta}+\left(m+\frac{M}{2}\right)gL\sin\theta=0$$

for small angles $\sin\theta=\theta$

$$\left(m+\frac{M}{3}\right)L^2\ddot{\theta}+\left(m+\frac{M}{2}\right)gL\theta=0$$

$$\omega_n=\sqrt{\frac{\left(m+\frac{M}{2}\right)g}{\left(m+\frac{M}{3}\right)L}}$$

Compound Pendulum problem



$$J_o\ddot{\theta} + mgd\theta = 0$$