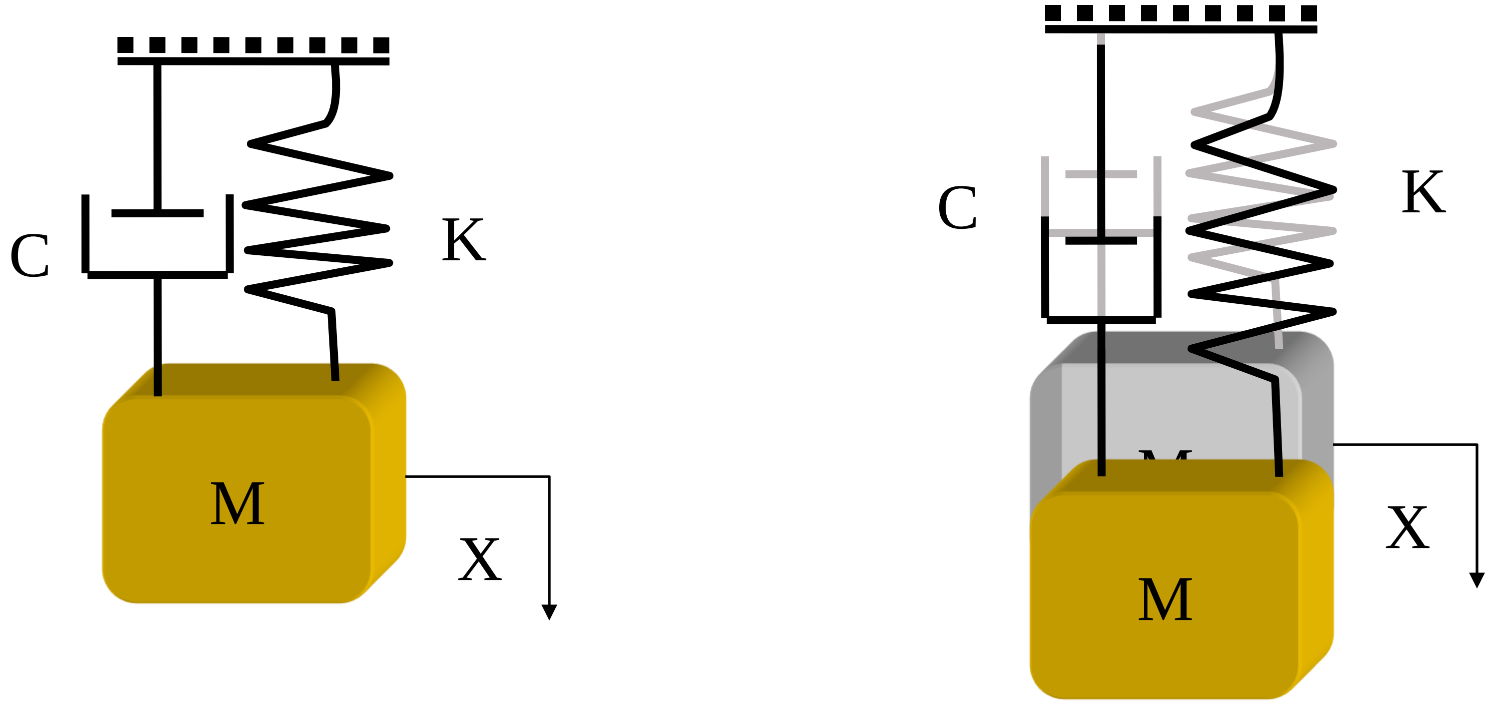
Single Degree of Freedom Damped Free Vibration



Given an initial condition, Determine the resulting motion

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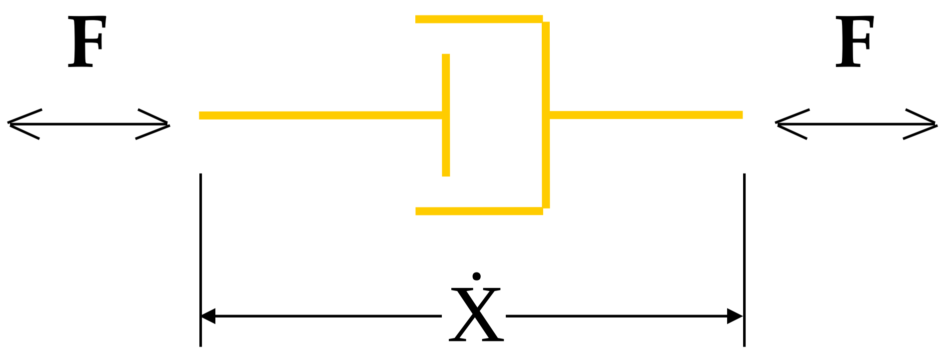
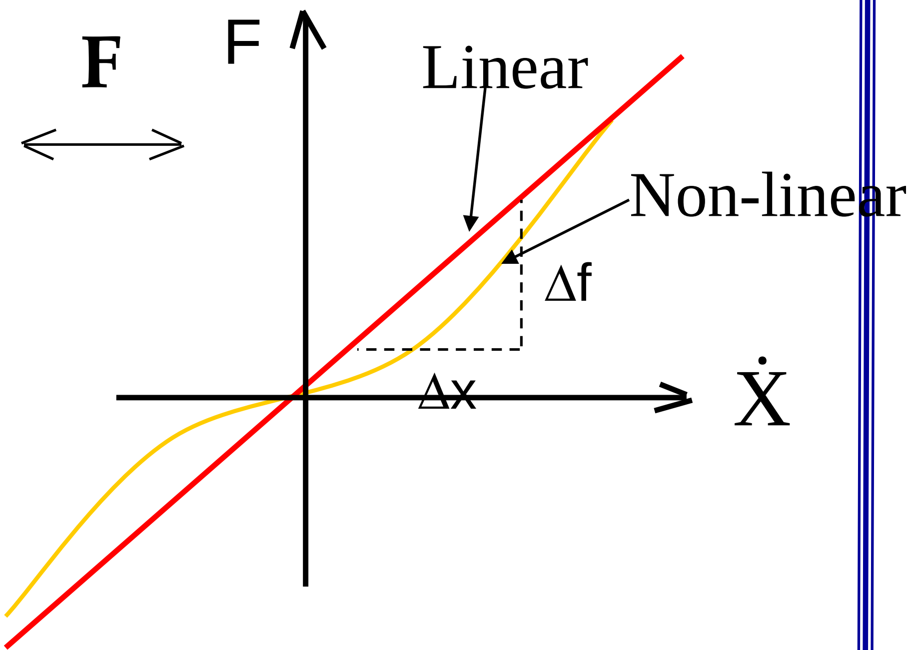
Viscous Damping Element (Dashpot)

- Damping Force is Linear and Proportional to Velocity.

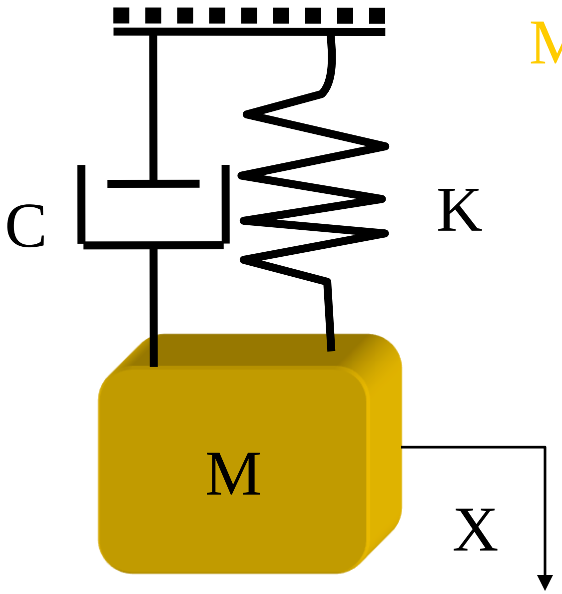




C is the viscous damping coefficient
units: N-sec/m or lbf-sec/in

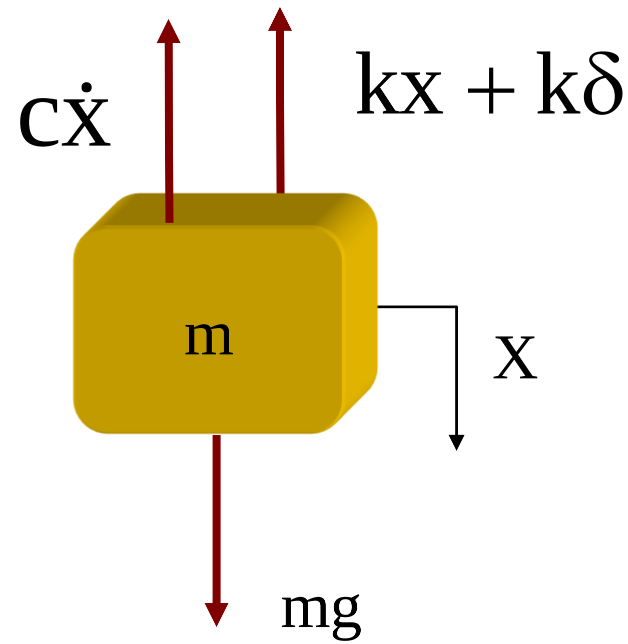
$$F = C\dot{X}$$

Maintain Dynamic Equilibrium



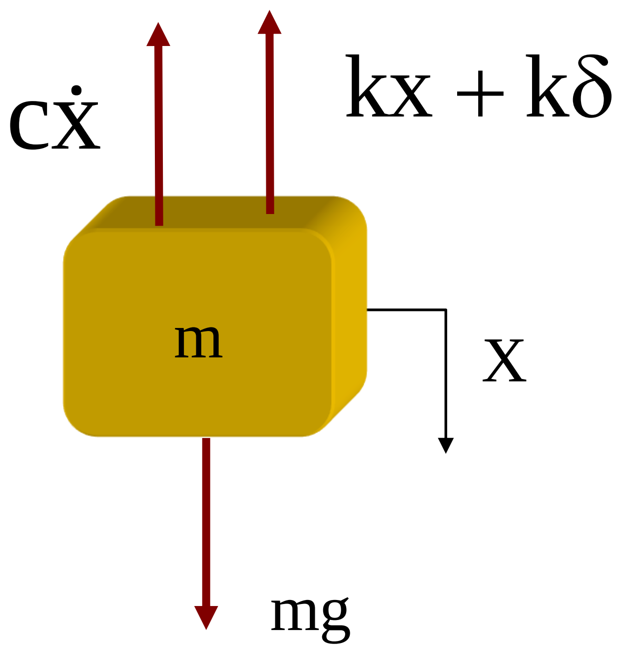
At rest, X= 0 (static equilibrium)

Free Body Diagram



Free Body Diagram

Maintain Dynamic Equilibrium



Apply Newton's 2nd Law

$$\sum F = m\ddot{x}$$

$$\sum F_{x\downarrow+} = m\ddot{x}$$

$$mg - (kx + k\delta) - c\dot{x} = m\ddot{x}$$

Equation of motion $\boxed{m\ddot{x} + c\dot{x} + kx = 0}$

Equation of motion $\boxed{m\ddot{x} + c\dot{x} + kx = 0}$

2nd order Differential equation

homogeneous

linear

Constant coefficients

Form of solution:

$$x(t) = X\sin(\omega t + \Phi) \quad \text{or} \quad x(t) = Ce^{st}$$

Equation of motion $\boxed{m\ddot{x} + c\dot{x} + kx = 0}$

$$\text{Assume} \quad x(t) = Ce^{st}$$

$$\dot{x}(t) = sCe^{st}$$

$$\ddot{x}(t) = s^2Ce^{st}$$

$$ms^2Ce^{st} + scCe^{st} + kCe^{st} = 0$$

$$(ms^2 + cs + k)Ce^{st} = 0$$

for a non - trivial solution

$$ms^2 + cs + k = 0$$

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Equation of motion $\boxed{m\ddot{x} + c\dot{x} + kx = 0}$

$$ms^2 + cs + k = 0$$

$$s_{1,2} = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

C_1 and C_2 are determined from initial conditions

$$s_{1,2} = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

Consider a case when, $c^2 - 4mk = 0$

Solving for c :

$$c = 2\sqrt{km} = C_c$$

C_c = critical damping

Define :

$$\omega_n = \sqrt{\frac{k}{m}} = \text{natural frequency}$$

$$\zeta = \frac{c}{C_c} = \text{damping ratio}$$

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta^2 - 1)}\,\omega_n$$

Case 1: $\zeta < 1$ Under damped (Complex conjugate roots)

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta^2 - 1)}\omega_n$$

$$s_{1,2} = -\zeta\omega_n \pm j\sqrt{(1-\zeta^2)}\omega_n$$

Define :

$\omega_d = \sqrt{(1-\zeta^2)}\omega_n =$ damped natural frequency

$$\boxed{s_{1,2} = -\zeta\omega_n \pm j\omega_d}$$

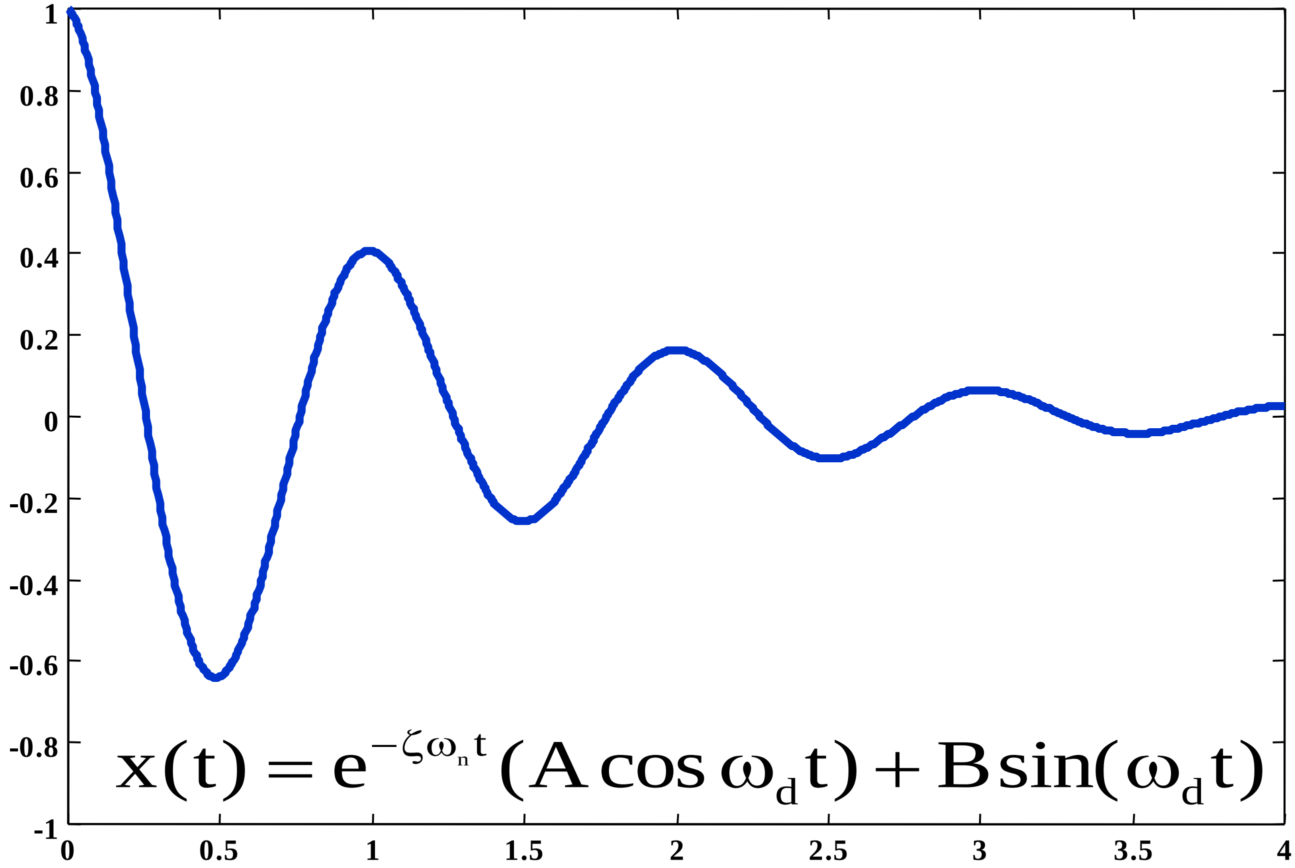
Case 1: $\zeta < 1$ Under damped (Complex conjugate roots)

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

$$x(t) = C_1 e^{(-\zeta\omega_n + j\omega_d)t} + C_2 e^{(-\zeta\omega_n - j\omega_d)t}$$

$$\boxed{x(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t) + B\sin(\omega_d t)}$$

Case 1: $\zeta < 1$ Under damped (Complex conjugate roots)



$$x(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t) + B\sin(\omega_d t)$$

Case 2: $\zeta = 1$ Critically damped (Real equal roots)

$$s_{1,2} = -\zeta\omega_n$$

$$s_1 = -\zeta\omega_n = -\sigma_1$$

$$s_2 = -\zeta\omega_n = -\sigma_2$$

$$x(t) = C_1 e^{-\sigma_1 t} + C_2 t e^{-\sigma_2 t}$$

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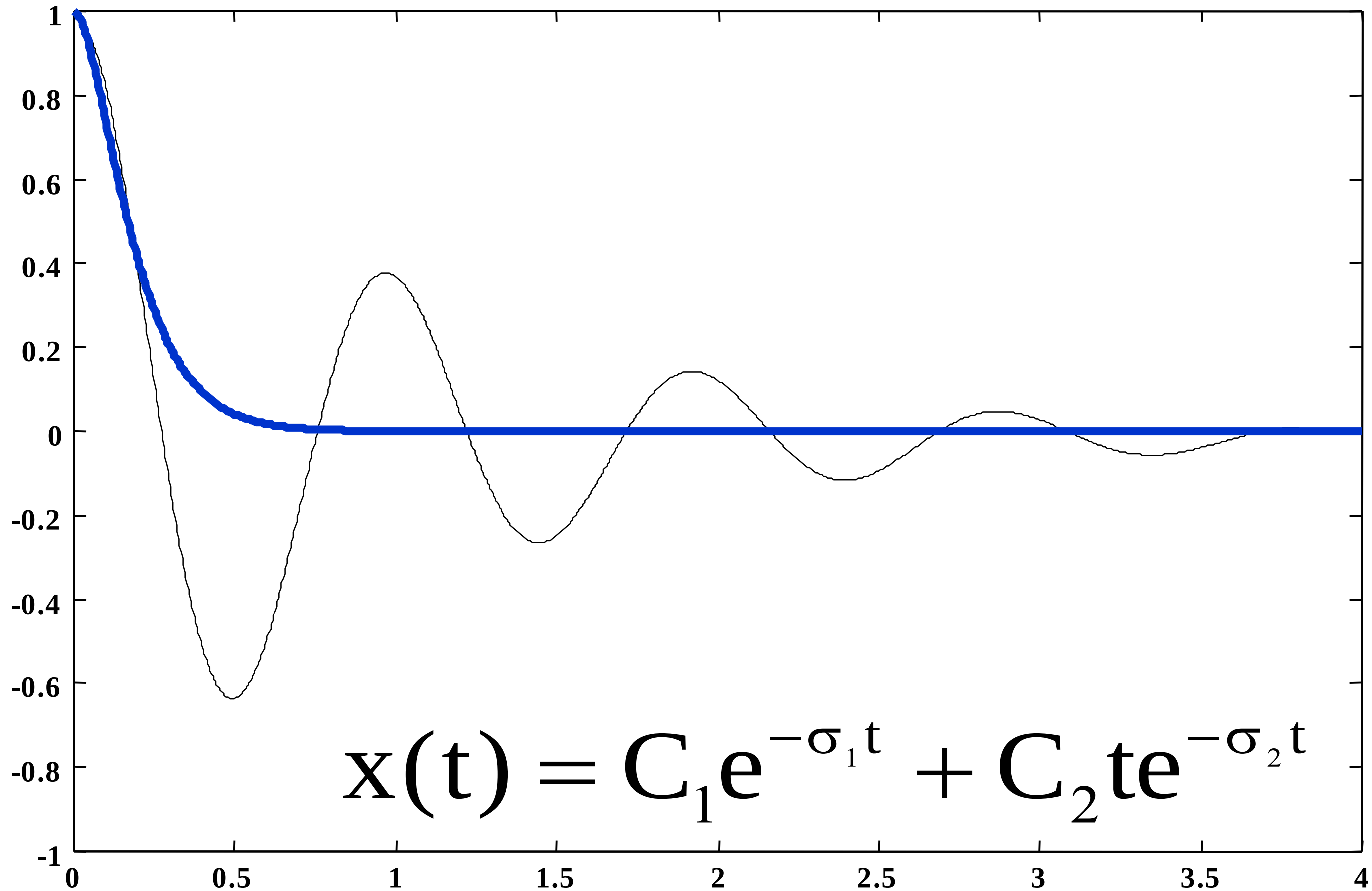
Case 2: $\zeta = 1$ Critically damped (Real equal roots)



$$x(t) = C_1 e^{-\sigma_1 t} + C_2 t e^{-\sigma_2 t}$$

Case 3: $\zeta > 1$ Over-damped (Real unequal roots)

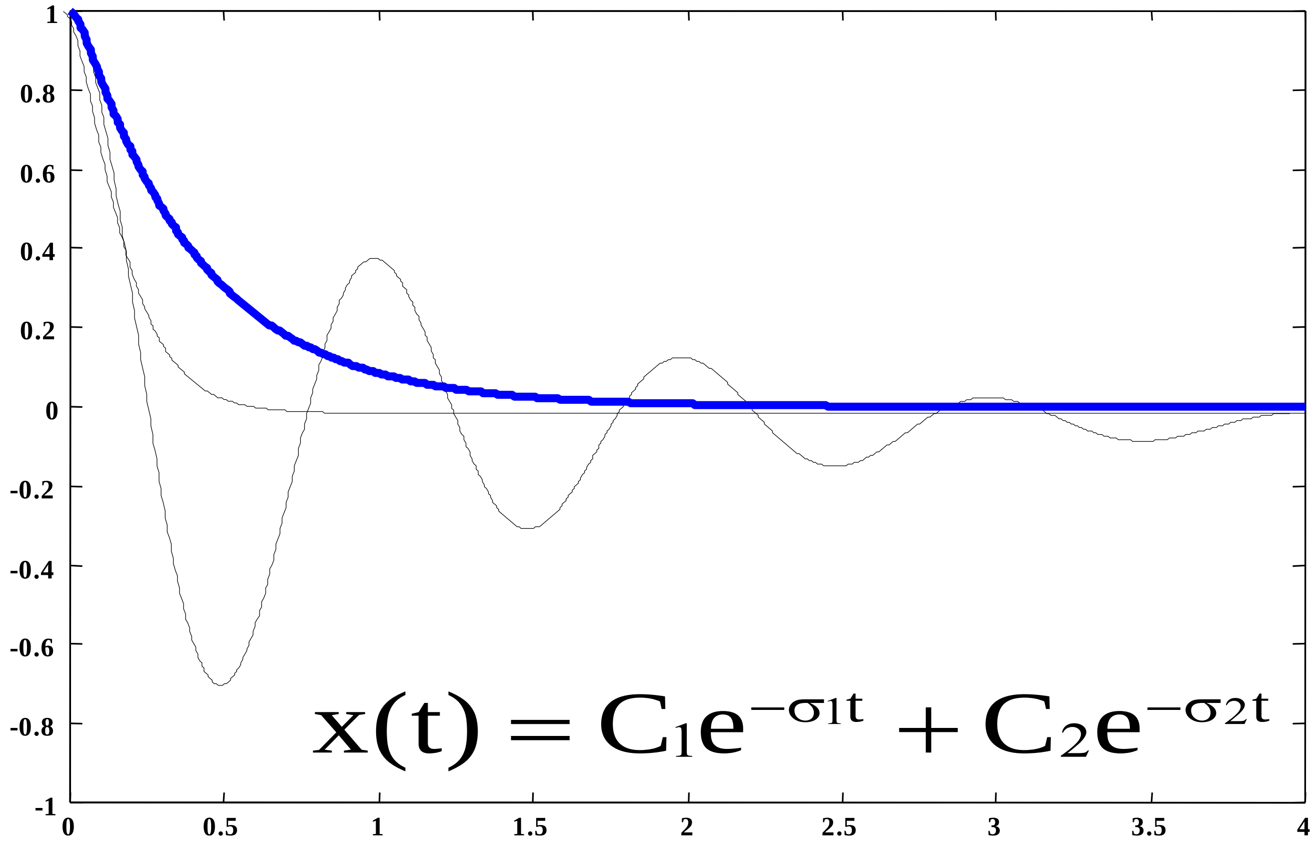
$$s_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta^2 - 1)}\omega_n$$

$$s_1 = -\zeta\omega_n + \sqrt{(\zeta^2 - 1)}\omega_n = -\sigma_1$$

$$s_2 = -\zeta\omega_n - \sqrt{(\zeta^2 - 1)}\omega_n = -\sigma_2$$

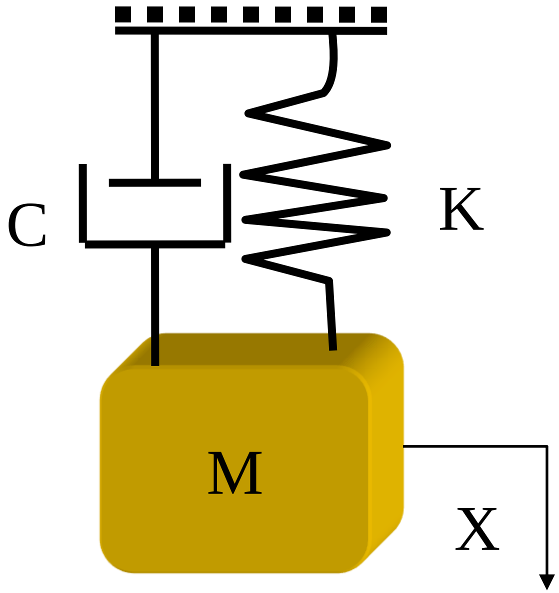
$$x(t) = C_1 e^{-\sigma_1 t} + C_2 e^{-\sigma_2 t}$$

Case 3: $\zeta > 1$ Over-damped (Real unequal roots)



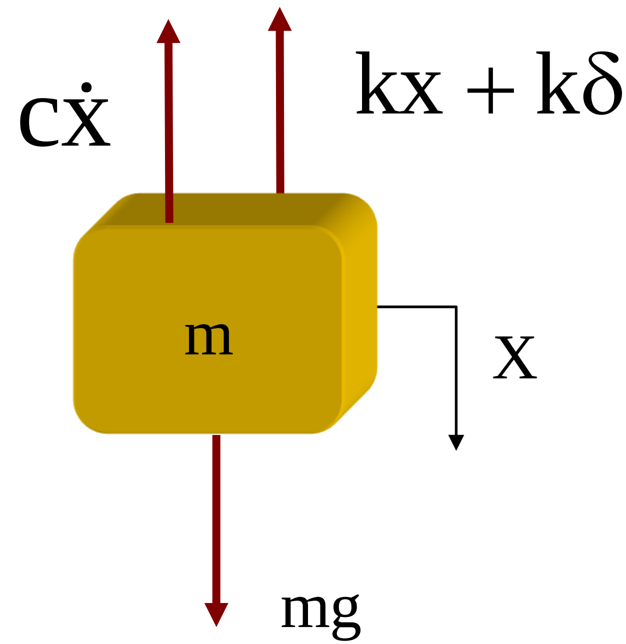
$$x(t) = C_1 e^{-\sigma_1 t} + C_2 e^{-\sigma_2 t}$$

Maintain Dynamic Equilibrium



At rest, X= 0 (static equilibrium)

Free Body Diagram



$$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

Case 1: $\zeta < 1$ Under-damped (Complex conjugate roots)

$$x(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

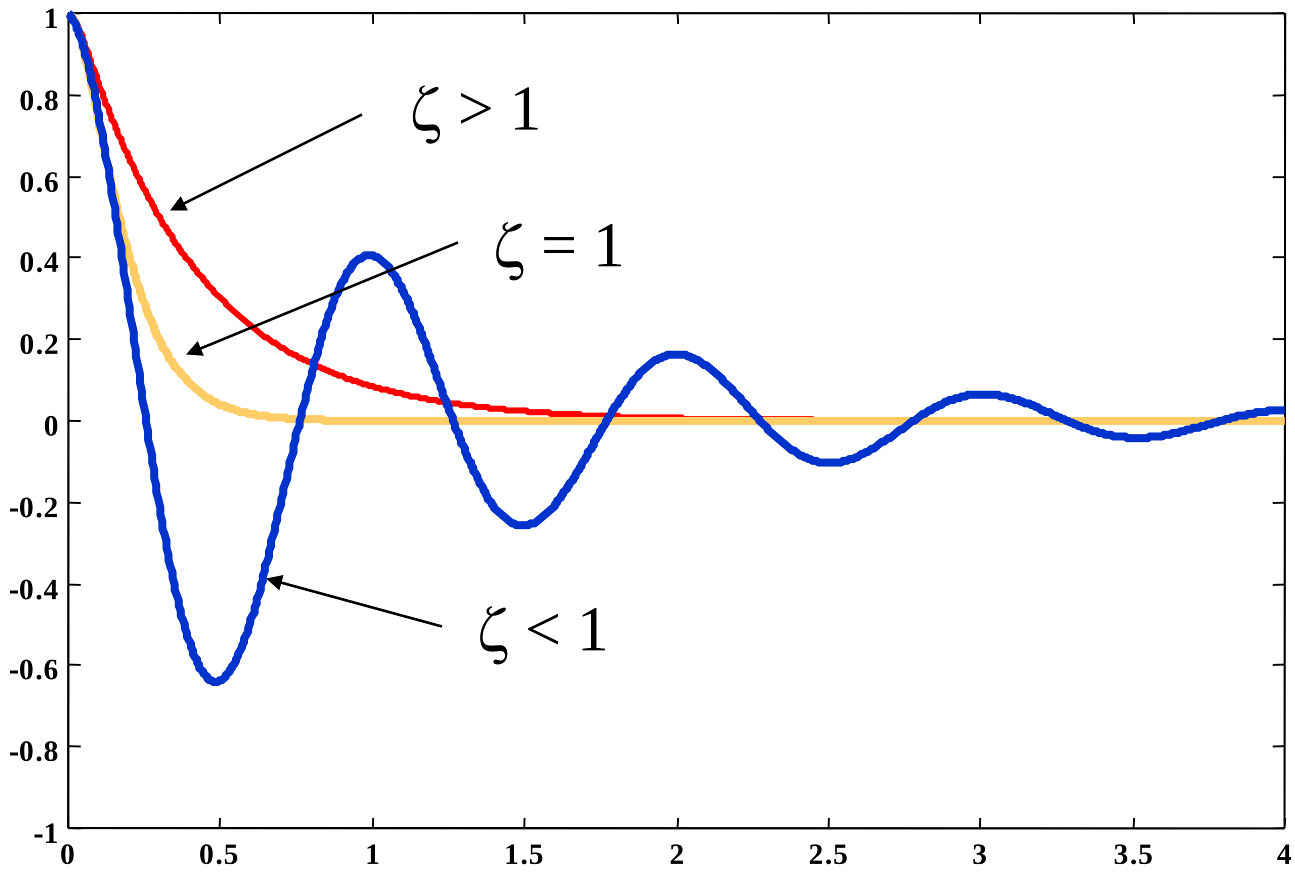
Case 2: $\zeta = 1$ Critically damped (Real equal roots)

$$x(t) = C_1 e^{-\zeta\omega_n t} + C_2 t e^{-\zeta\omega_n t}$$

Case 3: $\zeta > 1$ Over-damped (Real unequal roots)

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

Free Vibration Response: SDOF with Damping



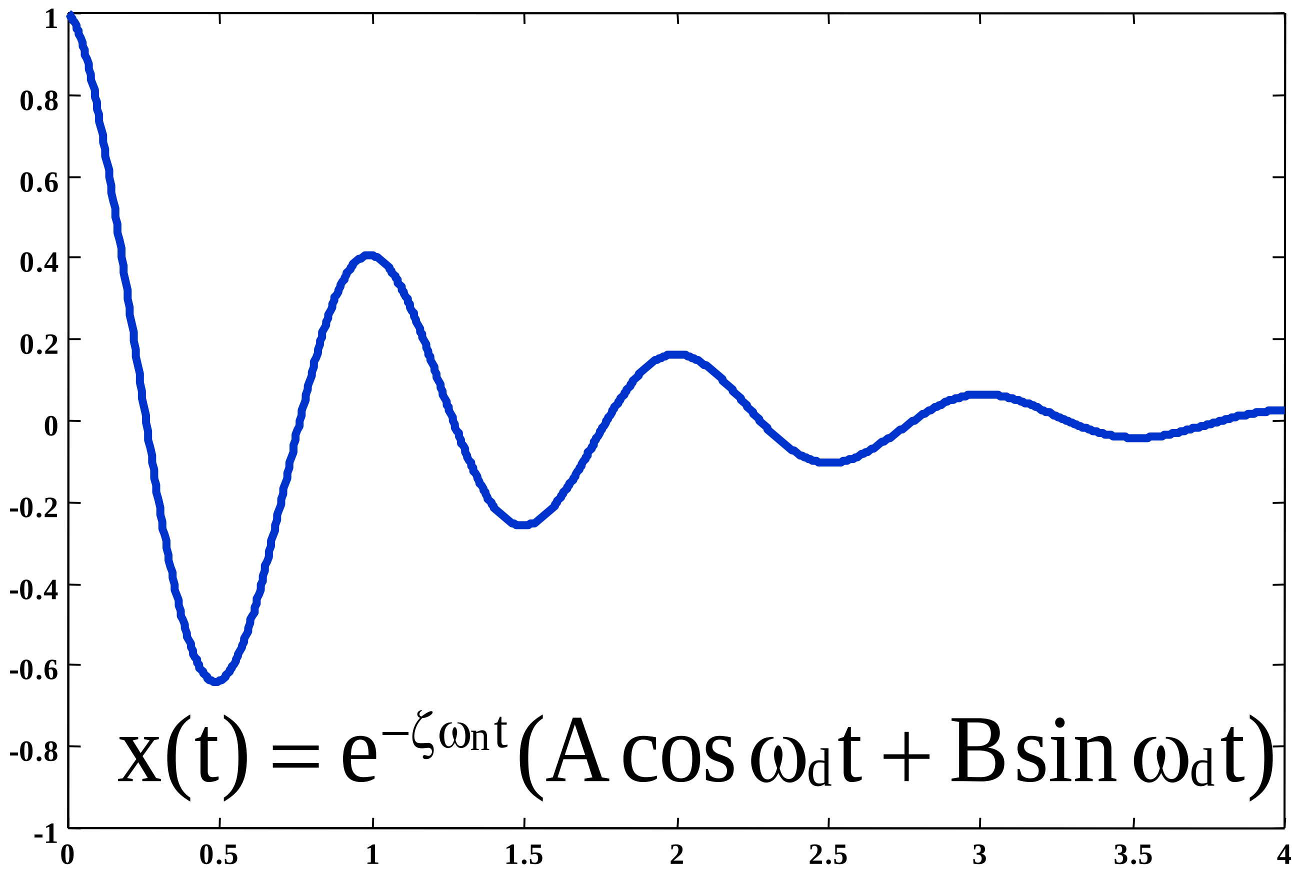
Damping ratio (for many structural materials) $0.001 \le \zeta \le 0.05$

% critical damping

$$\zeta * 100\%$$

$$0.1\% \le \zeta \le 5\%$$

Case 1: $\zeta < 1$ Under damped (Complex conjugate roots)



$$x(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

$$x(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

Determine A, and B from the given initial Conditions, x(0) and $\dot{x}(0)$

$$x(0) = \cancelto{1}{e^{-\zeta\omega_n 0}}\left(A\cos\left(\cancelto{1}{\omega_d 0}\right) + B\sin\left(\cancelto{0}{\omega_d 0}\right)\right)$$

$$x(0) = A$$

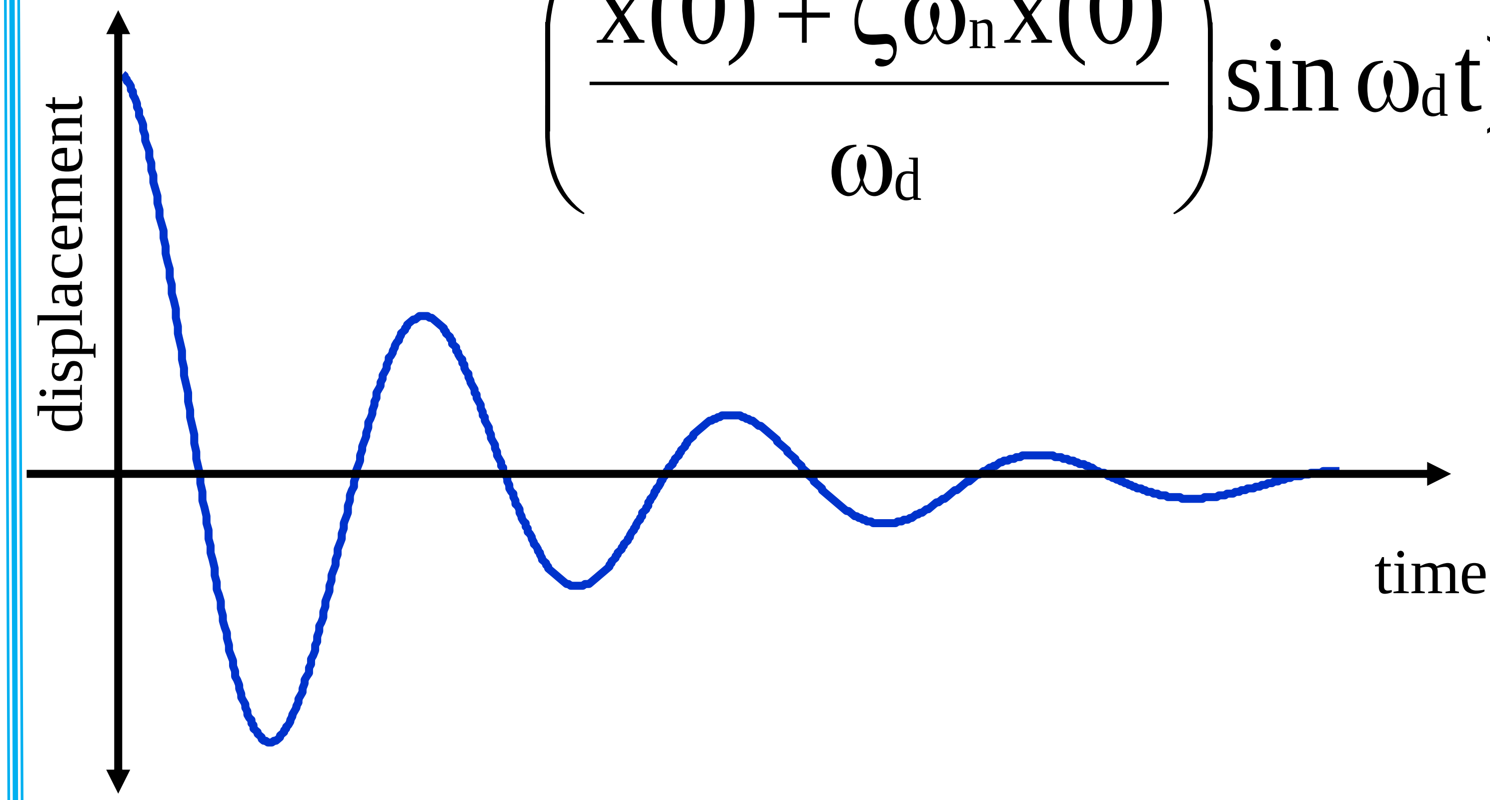
$$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}(x(0)\cos\omega_d t + B\sin\omega_d t) + e^{-\zeta\omega_n t}(-\omega_d x(0)\sin\omega_d t + \omega_d B\cos\omega_d t)$$

$$\dot{x}(0) = -\zeta\omega_n e^{-\zeta\omega_n 0}(x(0)\cos(\omega_d 0) + B\sin(\omega_d 0)) + e^{-\zeta\omega_n t}(-\omega_d x(0)\sin(\omega_d 0) + \omega_d B\cos(\omega_d 0))$$

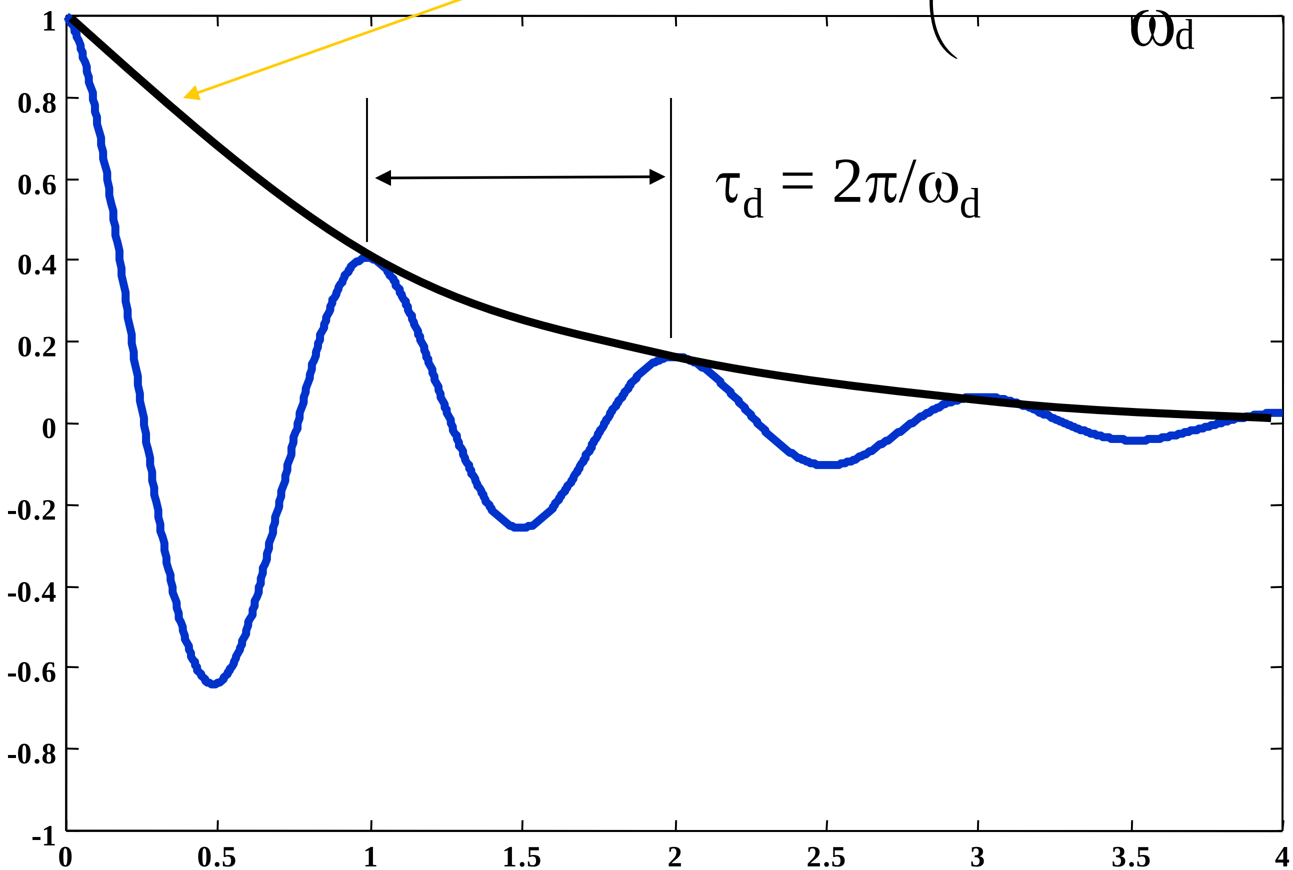
$$\dot{x}(0) = -\zeta\omega_n x(0) + \omega_d B$$

$$B = \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}$$

$$x(t) = e^{-\zeta\omega_n t}\left(x(0)\cos\omega_d t + \left(\frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}\right)\sin\omega_d t\right)$$



$$x(t) = e^{-\zeta\omega_n t}\left(x(0)\cos\omega_d t + \left(\frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}\right)\sin\omega_d t\right)$$



Alternate form of Equation of motion

$$\mathrm{m}\ddot{\mathrm{x}} + \mathrm{c}\dot{\mathrm{x}} + \mathrm{kx} = 0$$

$\omega_n = \sqrt{\dfrac{k}{m}} =$ natural frequency $\qquad \zeta = \dfrac{\mathrm{c}}{\mathrm{C_c}} =$ damping ratio

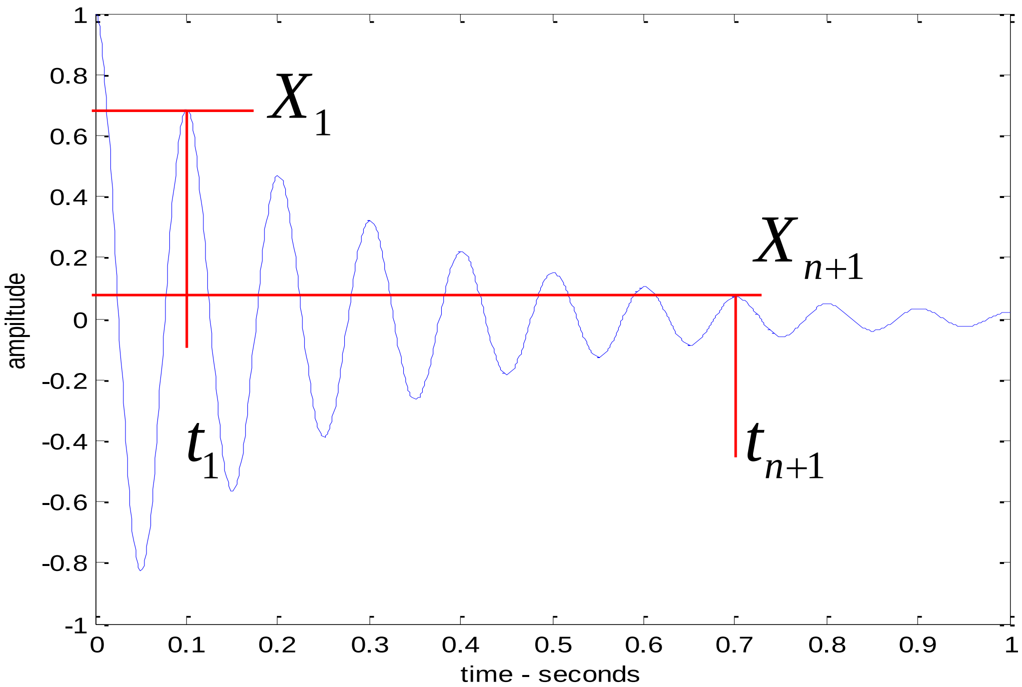
$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0}$$

Log Decrement

A method to experimentally measure damping ratio in under-damped systems



n = number of periods

$$t_{n+1} = t_1 + n\tau_D$$

$$\tau_D = \frac{2\pi}{\omega_d}$$

$$x(t) = e^{-\zeta\omega_n t}\left(X \sin\left(\omega_d t + \phi\right)\right)$$

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Log Decrement

$$\frac{x_1}{x_{n+1}} = \frac{e^{-\zeta\omega_n t_1}\left(X\sin\left(\omega_d t_1+\phi\right)\right)}{e^{-\zeta\omega_n t_{n+1}}\left(X\sin\left(\omega_d t_{n+1}+\phi\right)\right)}$$

$$\frac{x_1}{x_{n+1}} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n\left(t_1+n\tau_D\right)}} = e^{n\zeta\omega_n\tau_D}$$

Using: $\tau_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$

$\frac{x_1}{x_{n+1}}$ becomes

$$\frac{x_1}{x_{n+1}} = e^{\frac{n\zeta\omega_n 2\pi}{\omega_n\sqrt{1-\zeta^2}}} = e^{\frac{n2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

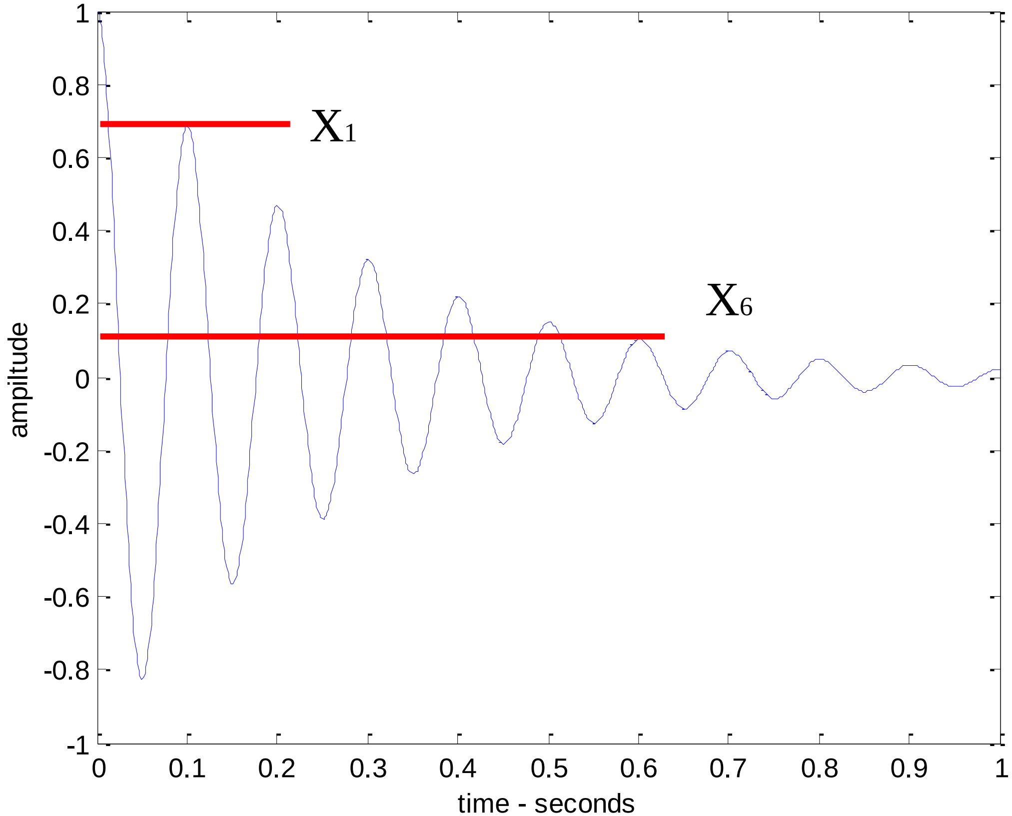
Log Decrement

Given → $$\frac{x_1}{x_{n+1}} = e^{\frac{n2\pi\zeta}{\sqrt{1-\zeta^2}}}$$ take ln ($\log_e$) both sides

$$\ln\left(\frac{x_1}{x_{n+1}}\right) = \frac{n2\pi\zeta}{\sqrt{1-\zeta^2}} \equiv$$ Log Decrement

$$\frac{1}{n}\ln\left(\frac{x_1}{x_{n+1}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\approx 2\pi\zeta \quad \text{for} \quad \zeta < 0.2$$



Example

$$X_1 = 0.68$$

$$X_6 = 0.12$$

$$\frac{1}{5}\ln\left(\frac{0.68}{0.12}\right) \approx 2\pi\zeta$$

$$\zeta = 0.055 \quad 5.5\%$$