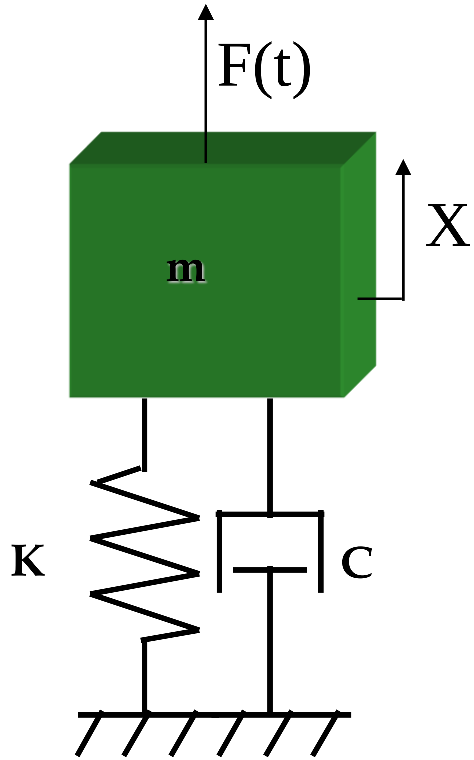
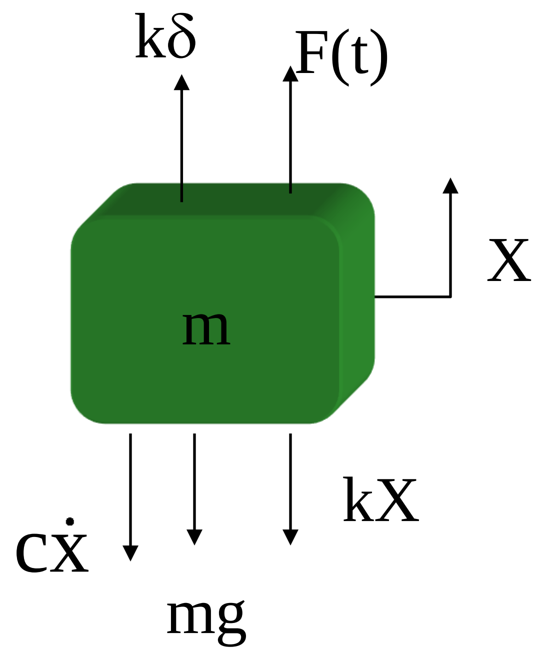
Single Degree of Freedom Forced Vibration



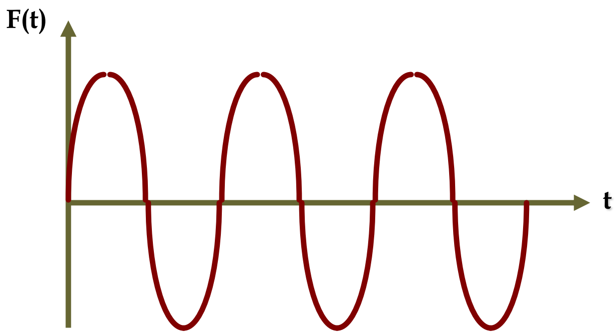
Free Body Diagram



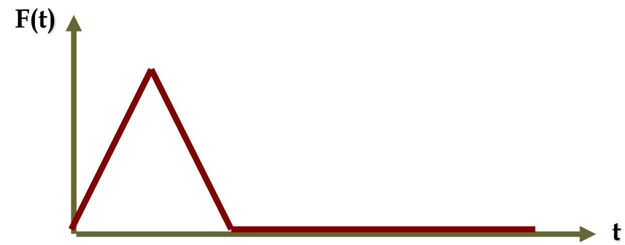
$$m\ddot{X} + c\dot{X} + kX = F(t)$$

There are 4 categories of F(t) :

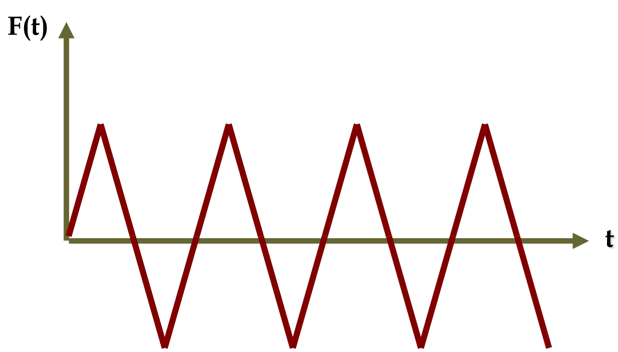
1) Harmonic (sin, cos)



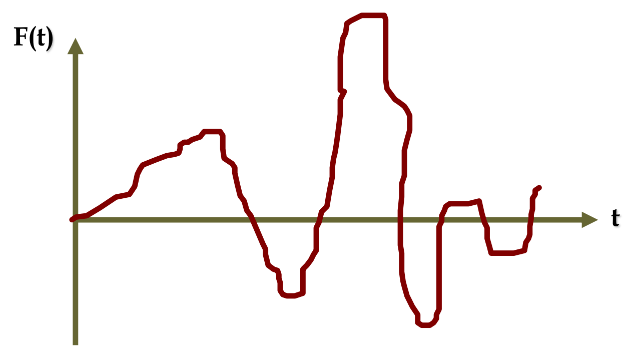
3) Transient



2) Periodic



4) Random

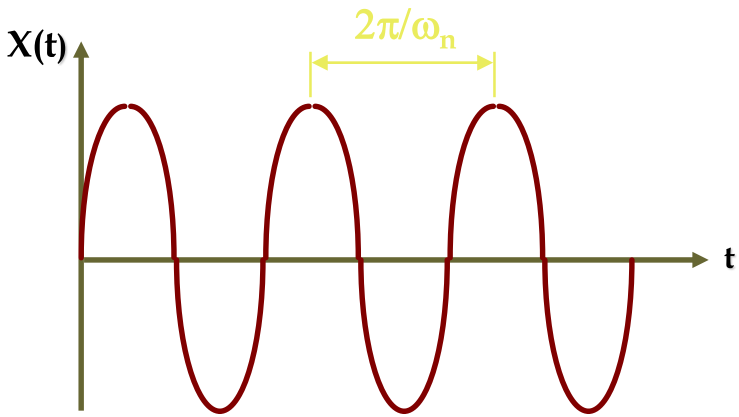


If $F(t) = F_o \cos(\omega t)$ ⟹ $m\ddot{X} + c\dot{X} + kX = F_o \cos(\omega t)$

If $c = 0$ ⟹ $m\ddot{X} + kX = F_o \cos(\omega t)$

$$X(t) = X_h(t) + X_p(t)$$

$$X(t) = X_{trans}(t) + X_{ss}(t)$$

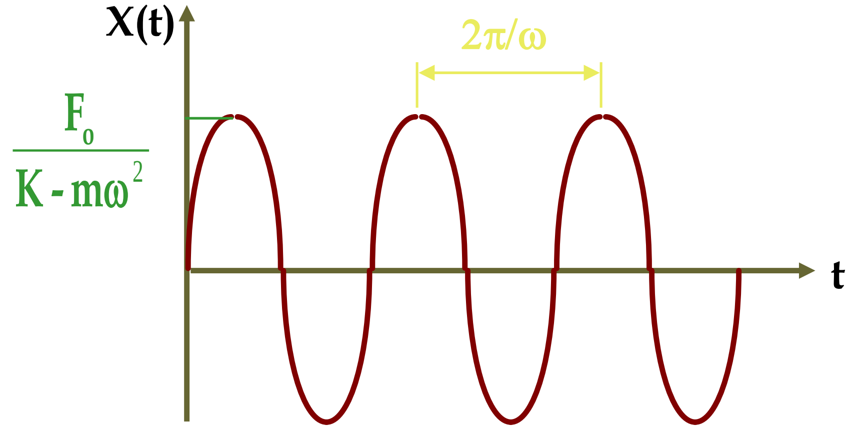


$$X_{trans}(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

$$X_{ss}(t) = X\cos(\omega t)$$

$$\dot{X}_{ss}(t) = -\omega X\sin(\omega t)$$

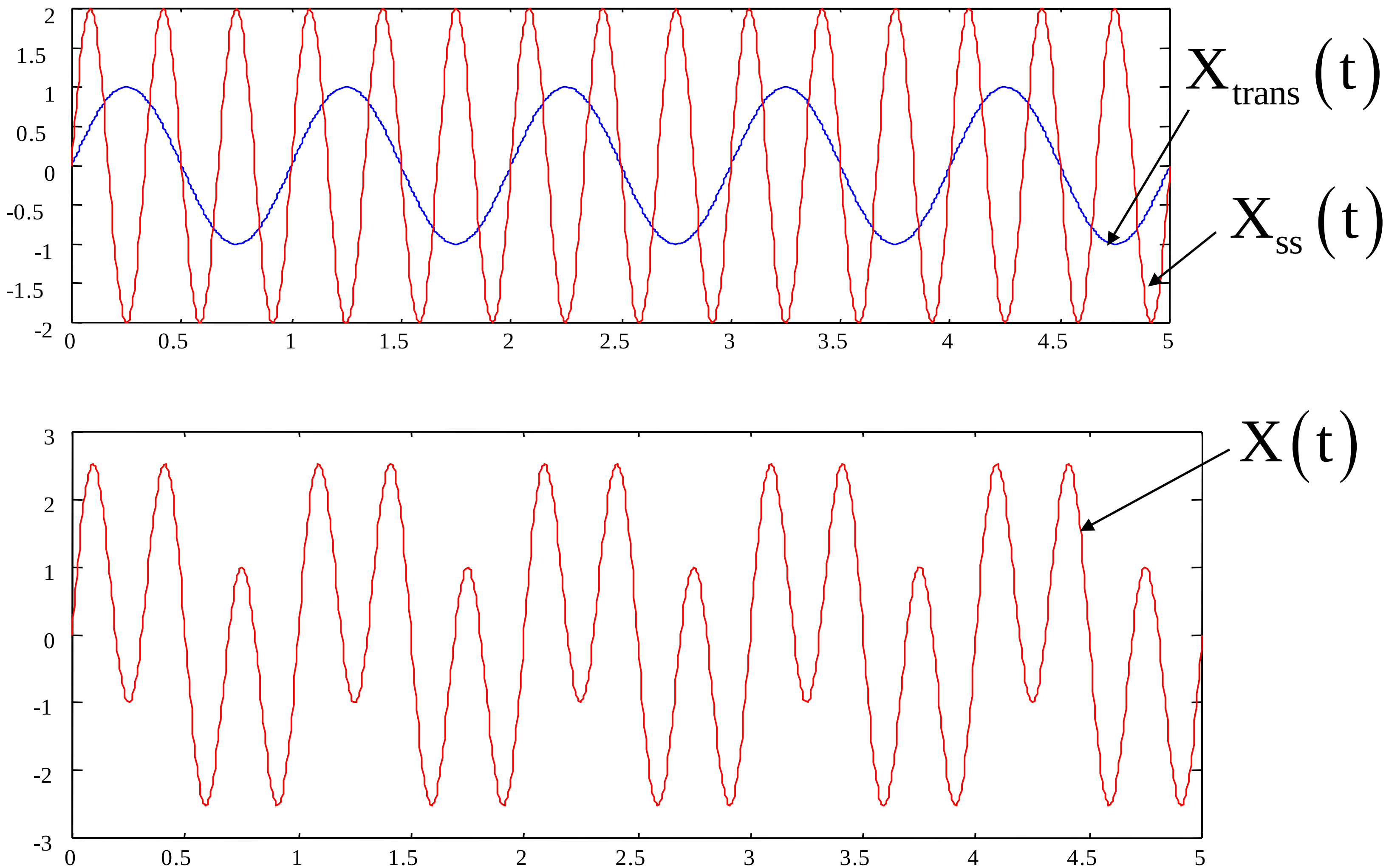
$$\ddot{X}_{ss}(t) = -\omega^2 X\cos(\omega t)$$



$$-m\omega^2 X\cos(\omega t) + kX\cos(\omega t) = F_o\cos(\omega t)$$

$$X = \frac{F_o}{k - m\omega^2}$$

$$X_{ss}(t) = \frac{F_o}{k - m\omega^2}\cos(\omega t)$$



2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
$X_{trans}(t)$
$X_{ss}(t)$
3
2
1
0
-1
-2
-3
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
$X(t)$

$$X(t) = A\cos(\omega_n t) + B\sin(\omega_n t) + \frac{F_0}{k - m\omega^2}\cos(\omega t)$$

$$X(0) = A + \frac{F_o}{K - m\omega^2} \Rightarrow A = X(0) - \frac{F_o}{K - m\omega^2}$$

$$\dot{X}(t) = -A\omega_n \sin(\omega_n t) + B\omega_n \cos(\omega_n t) - X\omega \sin(\omega t)$$

$$\dot{X}(0) = \omega_n B \Rightarrow B = \frac{\dot{X}(0)}{\omega_n}$$

$$X(t) = \left(X(0) - \frac{F_o}{k - m\omega^2} \right)\cos(\omega_n t) + \frac{\dot{X}(0)}{\omega_n}\sin(\omega_n t) + \frac{F_o}{k - m\omega^2}\cos(\omega t)$$

Case 1: *if* $\omega << \omega_n$

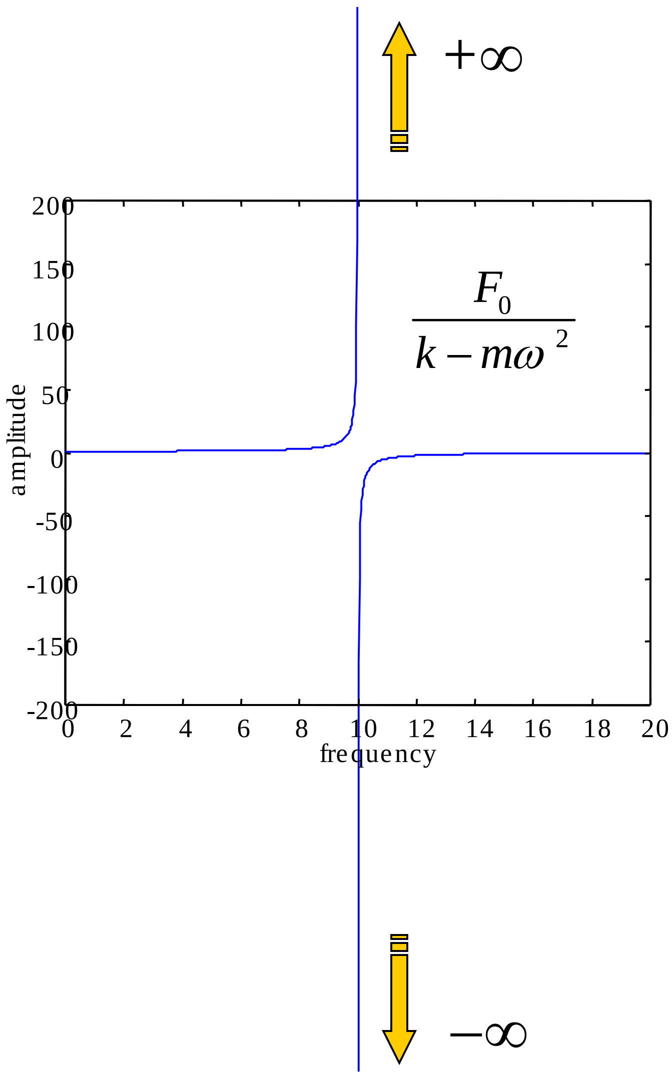
$$\frac{F_o}{k - m\omega^2} = \frac{F_o}{k - \frac{k\omega^2}{\omega_n^2}} \approx \frac{F_o}{k} = \delta_{st}$$

Case 2: *if* $\omega >> \omega_n$

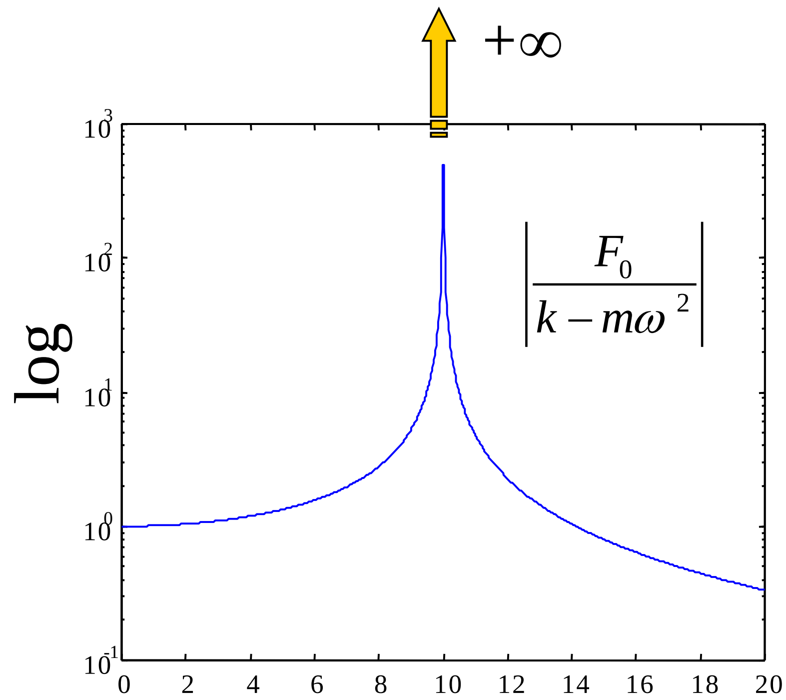
$$\frac{F_o}{k - m\omega^2} = \frac{F_o}{k - \frac{k\omega^2}{\omega_n^2}} \approx -\frac{F_o}{\frac{k\omega^2}{\omega_n^2}}$$

Case 3: *if* $\omega = \omega_n$

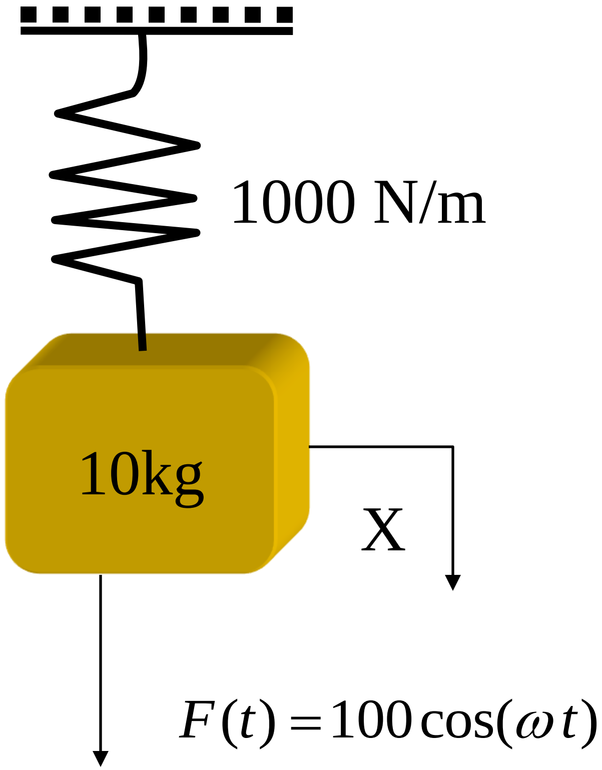
$$\frac{F_o}{k - m\omega^2} = \frac{F_o}{0} \approx \infty \quad \Rightarrow \quad X(t) = \frac{F_o \omega_n t}{2k} \sin(\omega_n t)$$



$+\infty$
200
150
100
50
0
-50
-100
-150
-200
amplitude
0 2 4 6 8 10 12 14 16 18 20
frequency
$\dfrac{F_0}{k - m\omega^2}$
$-\infty$



$+\infty$
log
10^3
10^2
10^1
10^0
10^{-1}
0 2 4 6 8 10 12 14 16 18 20
$\left|\dfrac{F_0}{k - m\omega^2}\right|$

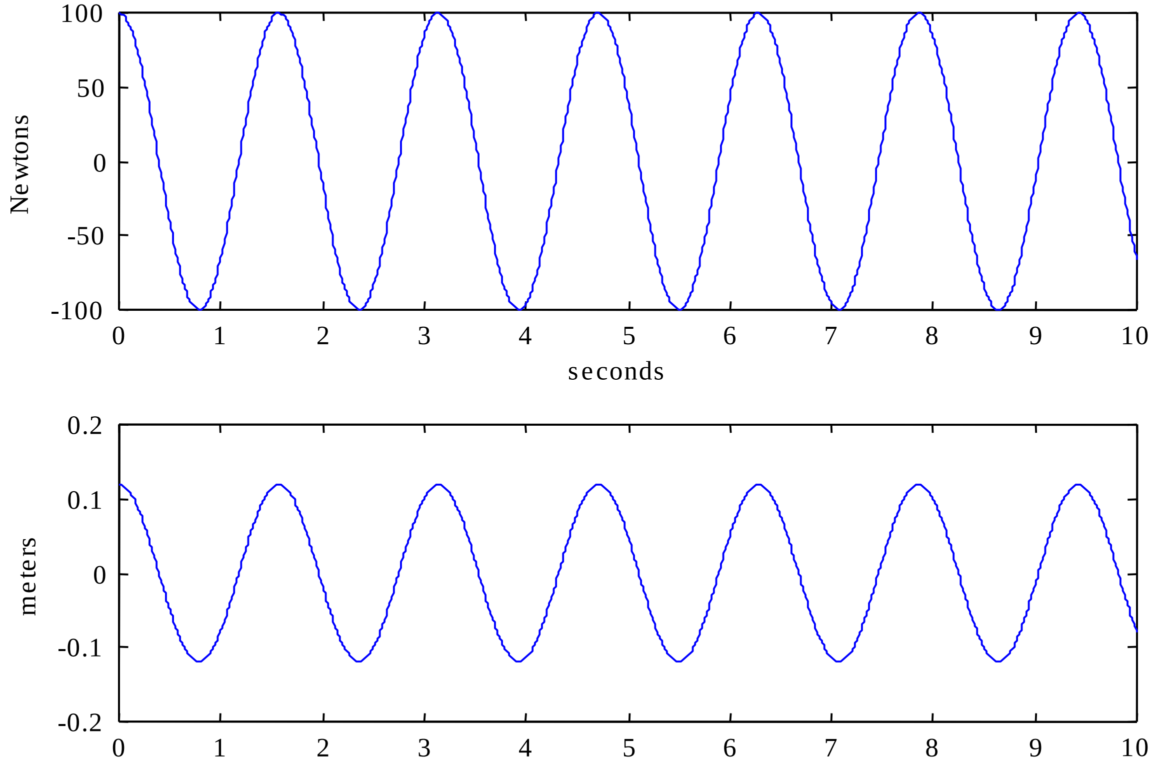


Steady State Response

$$x(t) = \frac{100}{1000 - 10\omega^2}\cos(\omega t)$$

$$\omega_n = \sqrt{\frac{1000}{10}} = 10 \text{ rad/sec}$$

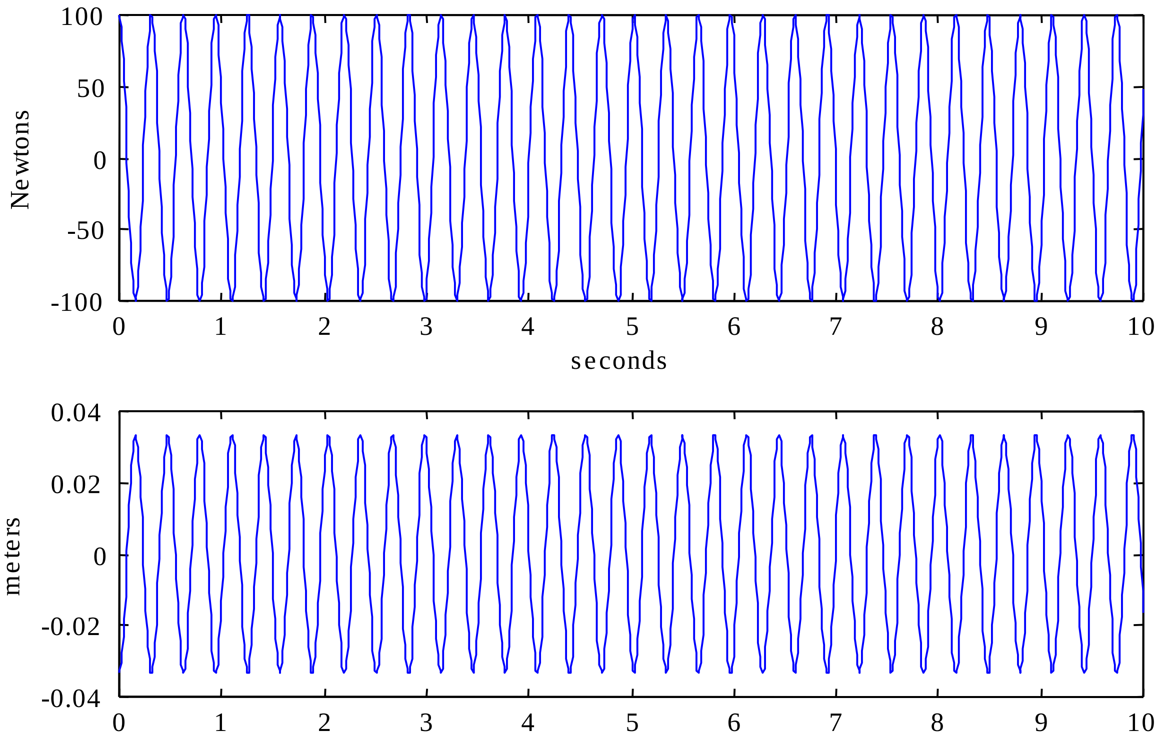
Case 1: *if* $\omega << \omega_n$



$$F(t) = 100\cos(4t)$$

$$X(t) = \frac{100}{840}\cos(4t)$$

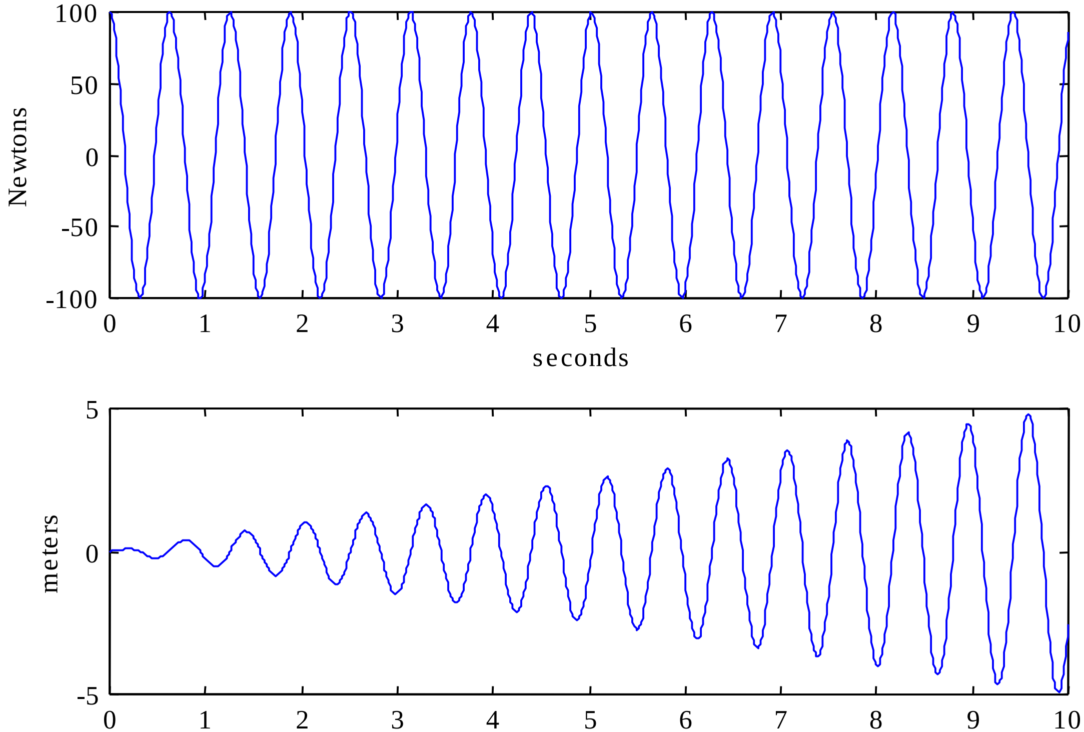
Case 2: *if* $\omega >> \omega_n$



$$F(t) = 100\cos(20t)$$

$$X(t) = -\frac{100}{3000}\cos(20t)$$

Case 3: *if* $\omega = \omega_n$



$$F(t) = 100\cos(10t)$$

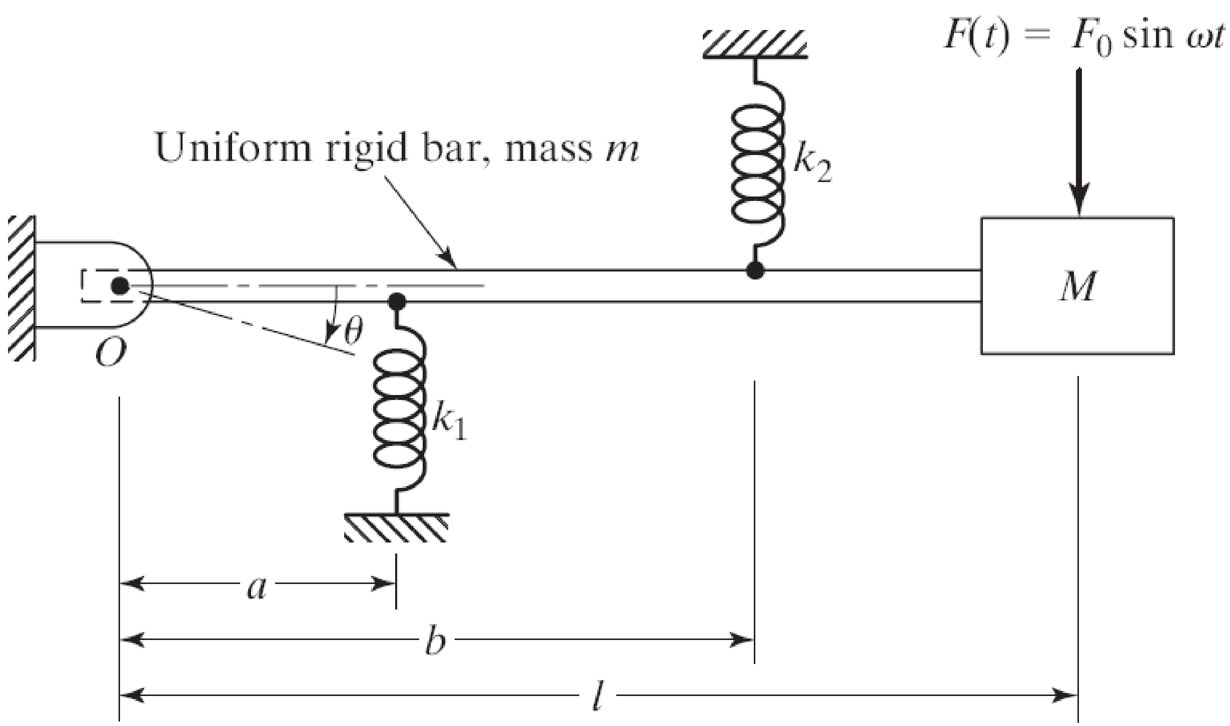
$$X(t) = \frac{1000t}{2000}\sin(10t)$$

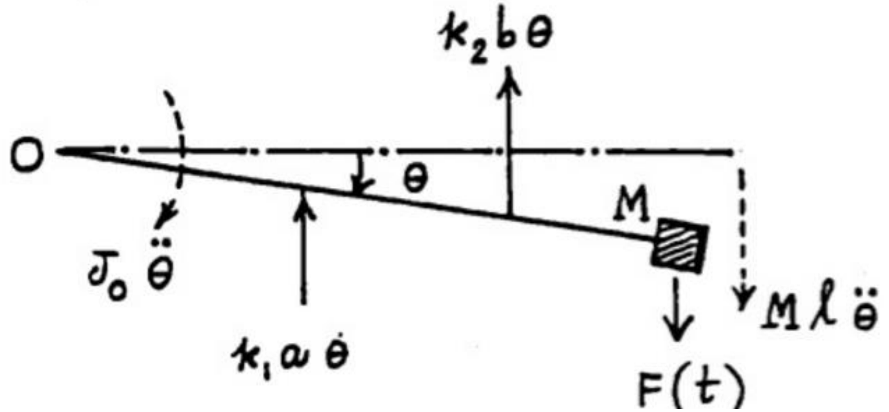
Example1:

A spring-mass system, with a spring stiffness of 5,000 N/m, is subjected to a harmonic force of magnitude 30 N and frequency 20 Hz. The mass is found to vibrate with an amplitude of 0.2 m. Assuming that vibration starts from rest ($x_0 = \dot{x}_0 = 0$), determine the mass of the system.

Example2:

Derive the equation of motion and find the steady-state response of the system shown in Fig. 3.44 for rotational motion about the hinge O for the following data: $k_1 = k_2 = 5000$ N/m, $a = 0.25$ m, $b = 0.5$ m, $l = 1$ m, $M = 50$ kg, $m = 10$ kg, $F_0 = 500$ N, $\omega = 1000$ rpm.





Equation of motion for rotational motion about the hinge O:

$$(J_0 + M\ell^2)\ddot{\theta} + (k_1 a^2 + k_2 b^2)\theta = F(t)\ell = F_0 \ell \sin \omega t \tag{1}$$

Steady state response (using Eqs. (3.3) and (3.6)):

$$\theta_p(t) = \Theta \sin \omega t \tag{2}$$

$$\text{where} \quad \Theta = \frac{F_0 \ell}{(k_1 a^2 + k_2 b^2) - (J_0 + M\ell^2)\omega^2} \tag{3}$$

$$\text{and} \quad J_0 = \frac{m\ell^2}{12} + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3} m \ell^2 \tag{4}$$

For given data, $J_0 = \frac{1}{3}(10)(1^2) = 3.3333$ kg–m^2, $\omega = \frac{1000\,(2\pi)}{60} = 104.72$ rad/sec,

$$\Theta = \frac{500\,(1)}{5000\,(0.25^2 + 0.5^2) - (3.3333 + 50\,(1^2))\,(104.72^2)} = -8.5718\,(10^{-4}) \text{ rad}$$